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Origin and Structure of VORTEX: Frontier of Nonlinear Science

渦の起源と構造：非線形科学の新展開

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A VORTEXE is a dynamic structure that is commonly observed in nature, society, as well as in cyber space, but, probably, it is the most difficult "thing" to identify, analyze, or even describe. To explore a VORTEX, one has to invoke some non-standard (e.g. nonlinear, non-canonical, non-commutative, non-Abelian, etc.) frameworks ---a theory of VORTEX will be constructed far beyond the conventional realm of physics, and *plasma physics* can be placed at the frontier of this challenge. The aim of this tutorial lecture is to show how the physics and mathematics of VORTEX is interesting, promising, and essential in various problems of plasma physics.

1. Introduction

Amongst various "structures" created in the Universe (as well as in society or in cyber space), VORTEX is the most common, fundamental, but still an enigmatic structure; the Universe is filled with vortexes (such as galaxies, accretion disks, stars and planetary systems, etc.) that clump and wind-up with magnetic fields (we will see that the fluid vortical motion is unified with magnetic fields). Strikingly absent in this rich narrative of growth and evolution of the cosmic systems, is a satisfactory "universal" identification and understanding of what a VORTEX is.

At the opposite pole to vortexes, there is another category of structures consisting of "waves" with oscillatory, solitary, or stepwise shapes. This rather well-understood group of structures is characterized by a *phase* (or an *action*). As far as a phase can be defined as a single-valued regular function, one may invoke the standard methods of mechanics (known as *Fermat's law* or the *least-action principle*). However, VORTEX is a fundamental breakdown of a simple phase description. An elementary form of vortex may be represented by an angular (multi-valued) phase, and then, the phase singularity defines a "point vortex." A general vortex no longer allows any definition of a phase, posing a challenge of geometrical or algebraic characterization.

After the "soliton" has took off from the mother land of plasma physics (and evolved into a sophisticated theory of algebraic analysis), we have had a long quiescence in the field of basic, general science. VORTEX is certainly a promising paradigm on which we may describe a new discourse of the Universe.

2. Vorticity in an Ideal Image

A fundamental difficulty of the standard mechanics in describing a vortex is epitomized in its very basic *Hamilton-Jacobi equation* $\boldsymbol{P} = \nabla \phi$ (ϕ is the action), which forces the momentum $\boldsymbol{P}$ to be vorticity-free ($\nabla \times \boldsymbol{P} = 0$). This is just fine as far as a single particle pertains. However, to describe a "fluid" (collection of an infinite number of orbits = stream lines), we have to introduce "non- canonicality" that allows $\boldsymbol{P}$ to have a finite vorticity.

A *non-canonical Hamiltonian system* [1] is formulated by a singular Poisson bracket that has a nontrivial kernel consisting of so-called *Casimir elements*, which foliates the phase space, i.e., the dynamics is constrained on a level-set of the Casimir element. The Casimir element pertinent to a vorticity is the *helicity*. A rich variety of structures are obtained on a foliated (constrained) phase space; even the equilibrium point (minimum of the Hamiltonian) has a nontrivial structure (the familiar "Taylor relaxed state" is one of them); we find an integrable (quantized) nonlinear torsional waves (Alfvén waves) near the equilibrium point [2].

3. Vortex and Entropy

The foregoing ideal framework, however, falls short to describe a story of "creation" of vorticity, since the invariance of the helicity poses a fundamental obstacle for a vorticity to emerge from zero. Here we delineate a basic relation between Kelvin's circulation law and the thermodynamic first-second laws, which, in the relativistic regime, provides us with a universal theory of "creation."

The *circulation* $\oint \delta Q$, associated with a variation δQ of a physical quantity may be zero or finite depending on whether δQ equals an exact differential dH of a state variable H or not. For example, if $\delta Q = TdS$ (T: temperature, S: entropy), the circulation is generally finite and measures the work done in a quasi-static thermodynamic cycle. An *ideal fluid* can be viewed as a realization of an infinite number of ideal isolated cycles; in the fluid-mechanical setting, however, the cycle must be evaluated along a *loop L* that moves with the fluid. And, to write a theory in the relativistic regime, we have to immerse the loop in the 4-d space-time and transport it by the 4-velocity $dx_\mu/ds = U_\mu$ where s denotes the proper time. The circulation pertinent to the theory of VORTEX is the loop integral of the canonical momentum $P^\mu = m^* c U^\mu + (q/c)A^\mu$, where $m^* c^2 = h$ is the molar enthalpy, q is the charge, and A^μ is the EM potential. We obtain

$$\frac{d}{ds}\left(\oint_{L(s)} P^\mu dx_\mu\right) = \oint_{L(s)} (\partial^\mu P^\nu - \partial^\nu P^\mu) U_\nu dx_\mu.$$

On the other hand, the equation of motion reads as $(\partial^\mu P^\nu - \partial^\nu P^\mu)U_\nu = \partial^\mu h - n^{-1}\partial^\mu p$. By the first and second laws, the right-hand side may be written as $T\partial^\mu S$. We thus have

$$\frac{d}{ds}\left(\oint_{L(s)} P^\mu dx_\mu\right) = \oint_{L(s)} T\partial^\mu S dx_\mu.$$

The rate of change of the circulation (left-hand side) parallels the work done $\oint \delta W$ (with $\oint dh \equiv 0$).

This generalized (relativistic) Kelvin's law can be read as an "antithesis" to the standard Kelvin's law that prohibits creation of circulation (vorticity) in a barotropic flow (TdS is an exact differential); writing the Lorentz-covariant circulation law on a reference frame, the vector components reads as

$$\frac{d}{dt}\left(\oint_{L(t)} \boldsymbol{P} \cdot d\boldsymbol{x}\right) = \oint_{L(t)} \gamma^{-1} T \nabla S \cdot d\boldsymbol{x}.$$

Even in a barotropic flow, the kinematic factor γ^{-1} makes the cycle non-exact and creates a circulation.

The space-time distortion inherent in relativistic dynamics, thus provides us with a mechanism that, by breaking the topological constraint forbidding the emergence of magnetic fields (vortexes), allows "general vorticities" ---naturally coupled vortexes of matter motion and magnetic fields--- to be created in an ideal fluid. The newly postulated relativistic mechanism, arising from the interaction between the inhomogeneous flow fields and inhomogeneous entropy, may be an attractive universal solution to the origin problem [3].

4. Thermal Equilibrium in Foliated Phase Space

Self-organization of a structure is, at its surface, an antithesis of the entropy *ansatz*. However, disorder can still develop at micro scale while a structure emerges on some macro scale; it seems more common in various nonlinear systems that order and disorder are simultaneous [4], and such coexistence may be possible if the self-organization and the entropy principle work on different scales. The self-organization of a magnetospheric plasma vortex is an example.

A systematic formulation of a scale hierarchy is given by a foliation of the kinetic phase space in terms of actions (adiabatic invariants). Interpreting an action μ (the magnetic moment, for instance) as the number of quantized "quasi-particles," we can construct a grand-canonical distribution function $f_\alpha \propto e^{-\beta H - \alpha\mu}$ (H: Hamiltonian, β: inverse temperature, α: chemical potential of the quasi-particle) that maximizes entropy on a macroscopic leaf where the microscopic action-angle variables are abstracted as quasi-particle numbers. While embedding the macroscopic leaf in the (laboratory-frame) total phase space, the transforming Jacobian weight forces an inhomogeneous density profile in the laboratory flat space (see Fig. 1).

An inhomogeneous magnetic field (typically a dipole) yields a strongly distorted invariant measure of the magnetized particles, creating a self- organized vortex with steep density gradient; this theory explains "inward diffusion" in magnetospheres, as well as their laboratory simulations [5].

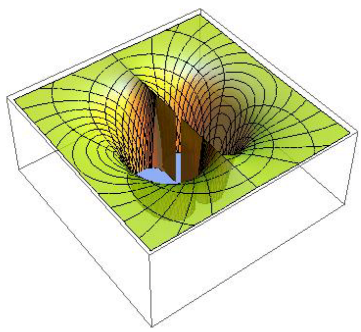

Fig.1. The distorted metric of foliated phase space of charged particles in a dipole magnetic field.

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