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Singularity analysis of two-dimensional two-fluid plasma
２次元２流体プラズマの特異性解析

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An essential difference of the plasma theory from the neutral fluid mechanics is in that plasma models may include a varity of "singular perturbations" determining scale hierarchies. Comparison of the ideal magnetohudrodynamics (MHD) and the two-fluid MHD may be the best practice by whch we can explore how a singular perturbation (electron inertia) determines an intrinsic (small) scale that is absent in the scale-invariant MHD system; in turn, how a sinlularity can emerge when an "obstacle" finite scale is removed with neglecting the singular perturbation. Here we study the role of Casimir invariants that characterize the "non-canonical" property of the determing symplectic geomerty; we define some different "sub-classes" of canonicalized self-contained mechanics of the reduced MHD, and report results of numerical analysis.

1. Introduction

Complex scale hierarchy is a fundamental characteristic of plasmas. Mathematically an *intrinsic scale* is determined by a "singular perturbation" that is represented by a term including a small scale parameter and a higher-order diffetential [1]. The ideal magnetohydrodynamics (MHD) is the base line that is free from any intrinsic scale, i.e. scale invariant. Including various singular perturbations, we obtain corresponding scales. Each intrinsic scale poses an obstacle preventing creation of singularity; this conjecture is still a tall order for rigorous proof (even for the conventional viscosity term that is a linear singular perturbation; the electron/ion inertia terms are nonlinear singular perturbations that yield dispersive effects, instead of dissipation). The aim of present study is to analyze scale hierarchy as a "phase-space foliation" determined by "sub-class dynamics" immersed in the total phase space.

2. Non-canonical Hamiltonian dynamics

The plasma fluid models have some conservation laws arising from either symmetries in the Hamiltonian or a "topological defect" (kernel) of the Poisson bracket [2]. The latter constants of motion is called Casimir invariants. A Hamiltonian system that has Casimir invariants is said "non-canonical." A Casimir invariant foliates the phase space, and the dynamics is constrained on a lieaf of the Casimir invariant.---as far as the phase space is of a finite dimension, this geometrical picture applies properly. However, in an infinite-dimension function space, the notion of Casimir leaf must be more carefully defined; if the Casimir invariant is not a continuous functional, even a bounded region on a Casimir leaf is not compact ---a lief (level-set) of such a Casimir is not such a smooth surface depict as a finite-dimension maniforld. We are interested in a possible pathological property created by a singularity of non-canonical Poisson bracket.

3. Reduced magnetohydrodynamics

P.J. Morrison and R.D. Hazeltine proposed a transformation of RMHD into a canonical form [4]. We denote the electrostatic potential by ϕ, and the flux function by ψ .The vorticity is $U=\nabla^2\phi$ and the current is $J=\nabla^2\psi$. Then the two-dimensional RMHD [3] are given by

$$\dot{U} = [\psi, J] + [U, \phi], \tag{1}$$
$$\dot{\psi} = [\psi, \phi], \tag{2}$$

where brackets are

$$[a, b] \equiv \mathbf{e}_z \cdot \nabla a \times \nabla b. \tag{3}$$

Pegoraro *et al.* showed that the dynamical system in the two-dimensional two-fluid model that contained electron inertia was represented as a non-canonical Hamiltonian form [5,6,7].

Morrison has introduced the following new variables.

$$\psi = [Q_1, Q_2], \quad U = [Q_1, P_1] + [Q_2, P_2] \tag{4}$$

Q_1, Q_2, P_1 and P_2 are new field variables introduced here. This change of variables gives the perspective from higher degrees of freedom.

4. Construction of subclasses

Morrison suggested that the transformation of these variables is not unique. We found several different kinds of transformation. According to non-canonical RMHD there exist following conserved Casimirs:

$$C_1 = \int \psi dxdy, \quad (5)$$
$$C_2 = \int h(\psi) Udxdy, \quad (6)$$

where $h(\psi)$ is an arbitrary function of ψ. We can consider another transformation:

$$\psi = \frac{1}{2}(Q_1^2 + Q_2^2), \ \ U = [Q_1, P_1] + [Q_2, P_2]. \quad (7)$$

The characteristic of this transformation is that C_2 becomes zero. It is easy to check the following:

$$C_2 = \int \psi Udxdy = 0. \quad (8)$$

This transformation increases the degrees of freedom.

The following transformation maintains two variables, which allows us writing down the canonical Hamiltonian form.

$$\psi = Q^2, \ \ U = [Q, P] \quad (9)$$

RMHD becomes:

$$D_t Q = 0, \quad (10)$$
$$D_t P = 2JQ. \quad (11)$$

D_t is the convective derivative, $D_t = \partial_t + [\phi, \ ]$. C_2 is equal to zero. We can analyze subclasses without increasing the degrees of freedom.

5. Numerical experiments

In order to analyze RMHD systems, we built a simulation code. We performed preliminary numerical experiments to verify whether the code is correct or not. The finite difference method was used to develop code for the spatial second order accuracy. Time is advanced using third order explicit Adams-Bashforts method. The boundary conditions are periodic.

The analytical solution is compared to numerical solution. We assume that $\psi_0 = \psi_0(x) = ax$, $\phi_0 = 0$ where ψ_0 and ϕ_0 are the equilibrium fields. We linearize the system

$$\psi = \psi_0 + \tilde{\psi}, \quad \phi = \phi_0 + \tilde{\phi}.$$

We also neglect x dependence, then equations become

$$\partial_t \tilde{\phi}'' = -a\tilde{\psi}''', \quad (12)$$
$$\partial_t \tilde{\psi} = -a\tilde{\phi}'. \quad (13)$$

The prime denotes the derivative of a function with respect to y. These are the well-known wave equations. We have observed the wave motion to proceed in the direction of y. This result is consistent with the analytical solution.

We began with the initial conditions:

$$Q = \sin\left(\frac{2\pi x}{10}\right),$$
$$P = \exp[-(x-5)^2 - (y-5)^2].$$

Now all variables are normalized. Fig.1 shows the vorticity of the fluid when $t = 5$. It can be seen that the small vortex occurs continuously. We observe gradually sharpening vorticity.

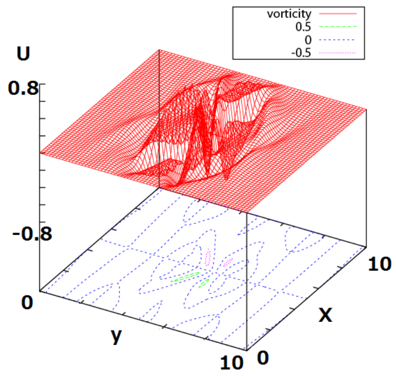

Fig.1. Vorticity

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