Approximate Formulae Representing Effective-Charge-Number Dependence of Collisional Relaxation Coefficients for Electron Fluid Moment Equations in Open Magnetic Field System

Tomonori TAKIZUKA[1)]*, Satoshi TOGO[2)]

[1)] TFuEL, Tokai-mura 319-1112, Japan
[2)] Plasma Research Center, University of Tsukuba, Tsukuba 305-8577, Japan

(Received 15 March 2026 / Accepted 12 May 2026)

We study the collisional relaxation processes of electron fluid moments in open magnetic field systems, such as scrape-off layer and divertor plasmas, by using a simplified Fokker-Planck equation. Anisotropies in temperature and heat flux in directions parallel and perpendicular to the magnetic field are taken into account. The thermal force is not given by the temperature gradient but by the heat flux damping. The relaxation of heat-flux anisotropy is found to play a role in the heat flux transport. For the convenience of plasma fluid simulations, approximate formulae are devised to represent the dependence of relaxation coefficients on the effective charge number.

Keywords: plasma fluid simulation, SOL divertor, moment equation, collisional relaxation, electron heat flux, thermal force, anisotropic temperature, effective charge number

DOI: 10.1585/pfr.21.1403040

1. Introduction

Heat transports in peripheral plasmas of magnetic confinement systems have been investigated by using plasma fluid models [1–3]. Magnetic field lines in the scrape-off layer (SOL) plasma hit directly plasma facing components, such as divertor plates. The outflowing power from a core plasma is transported mainly along the open magnetic field $\boldsymbol{B}$ in the SOL-divertor plasma. Major channel of the heat transport along $\boldsymbol{B}$ is the electron heat conduction, which is governed by Coulomb collisions. In the general electron fluid equations, the parallel conductive heat flux is given by $q_\mathrm{e} = -\kappa_{\|\mathrm{e}} \nabla_\| T_\mathrm{e}$. The thermal conductivity is defined as $\kappa_{\|\mathrm{e}} = \gamma_0 n_\mathrm{e} T_\mathrm{e} \tau_{\mathrm{e}0} / m_\mathrm{e}$, where the subscript $\|$ denotes the parallel to $\boldsymbol{B}$, T_e is the electron temperature, n_e the electron density and m_e the electron mass. The electron collision time $\tau_{\mathrm{e}0}$ is proportional to $T_\mathrm{e}{}^{3/2}$ and inversely proportional to the effective charge number of background ions Z_eff as well as to n_e; $\tau_{\mathrm{e}0}$ = 3.5 × 10^{11} $T_\mathrm{E}{}^{3/2} / n_\mathrm{e} Z_\mathrm{eff} \Lambda$ ($\tau_{\mathrm{e}0}$: s, T_E: eV, n_e: m^{-3}, Λ: Coulomb logarithm) [4]. A constant called "thermal-conductivity coefficient" is γ_0 = 3.16 for Z_eff = 1, and is increased gradually with Z_eff.

The friction force, $F_\mathrm{fr} = -\alpha_0 n_\mathrm{e} m_\mathrm{e} U / \tau_{\mathrm{e}0}$, and the thermal force, $F_\mathrm{th} = -\beta_0 n_\mathrm{e} \nabla_\| T_\mathrm{e}$, are also dependent on Z_eff nonlinearly (U: electron mean flow velocity relative to ion flow). A constant called "friction-force coefficient" α_0 = 0.51 for Z_eff = 1 is decreased gradually with Z_eff, while another constant called "thermal-force coefficient" β_0 = 0.71 for Z_eff = 1 is increased gradually with Z_eff [4]. For the convenience of simulation studies, we devise approximate formulae representing the Z_eff dependence of these relaxation coefficients.

In Sec. 2, a simplified Fokker-Planck equation is applied to study the collisional relaxation processes of an electron fluid in an open magnetic field system, such as SOL-divertor plasma. The anisotropy in T_e and q_e in directions parallel and perpendicular to $\boldsymbol{B}$ is taken into account. Relaxation terms for fluid moment equations are derived in Sec. 3. Approximate formulae representing the Z_eff dependence of relaxation coefficients are devised. Summary and discussion are given in Sec. 4.

*Corresponding author's e-mail: takizuka.tomonori@gmail.com
†DBA name, TFuEL, comes from "Takizuka Fusion Energy Lab"

2. Basics of Fluid Moment Equations

We derive electron fluid moment equations in an open magnetic field system, where the transport along $\boldsymbol{B}$ is dominant. Five fluid moments are taken into account in this paper, a mean flow (or electric current), two anisotropic temperatures (or effective isotropic temperature and temperature anisotropy), and two anisotropic heat fluxes (or total heat flux and heat flux anisotropy). Relaxation terms are derived by taking account of a simplified Fokker-Planck equation.

2.1 Distortion of velocity distribution function

Consider a plasma in a uniform magnetic field $\boldsymbol{B}$. Electrons are perturbed from a thermal equilibrium state. Mean flow, anisotropic temperature and heat flux are the results of the perturbation. The velocity distribution function of electrons is distorted a little from a Maxwellian, $f_M = n_e(m_e/2\pi T_e)^{3/2}$ exp $(-m_e v^2/2T_e)$, and is expressed as

$$f_e = f_M + \delta f_e = f_M + (\delta f_U + \delta f_{\Delta T} + \delta f_q). \quad (1)$$

The "mean-flow (MF) distortion" δf_U and the "anisotropic-temperature (AT) distortion" $\delta f_{\Delta T}$ are derived by expanding a shifted bi-Maxwellian distribution, $f_{SBM} \sim \exp\{-m_e (v_\parallel - U)^2/2T_{e\parallel}\} \times \exp\{-m_e v_\perp^2/2T_{e\perp}\}$,

$$\delta f_U = (zvU/v_{te}^2)\mathrm{G}_U f_M, \quad (2)$$

$$\delta f_{\Delta T} = \Delta_T(3z^2 - 1)(v^2/4v_{te}^2)f_M, \quad (3)$$

where v is the absolute velocity, $z = \cos\theta$ is a pitch angle variable ($\theta = 0$ is at the direction of $\boldsymbol{B}$) and $v_{te} \equiv (T_e/m_e)^{1/2}$ is the electron thermal speed. The parallel velocity is $v_\parallel = vz$, and the perpendicular velocity $v_\perp = v(1 - z^2)^{1/2}$ (subscripts $\parallel$ and $\perp$ denote the components parallel and perpendicular to $\boldsymbol{B}$, respectively).

Fluid moments are obtained by integrating f_e in the velocity space, $\boldsymbol{M}^{(n)} = \langle \boldsymbol{v}^n \rangle = (1/n_e)\int d\boldsymbol{v}(\boldsymbol{v}^n f_e) = (2\pi/n_e) \int\int dz v^2 dv(\boldsymbol{v}^n f_e)$. The 1st order moment is the electron mean flow velocity U relative to the ion flow velocity, which comes from a MF distortion δf_U; $U = \langle v_\parallel \rangle = (1/n_e)\int d\boldsymbol{v}(v_\parallel \delta f_U)$. In the parallel direction to $\boldsymbol{B}$, a shifted Maxwellian distribution is further deformed by the competitive relaxation due to electron-ion (e/i) and electron-electron (e/e) collisions. We conveniently put an adjustment function in Eq. (2), $\mathrm{G}_U = \mathrm{G}_U(\xi, Z_{eff})$ with $\xi \equiv v/v_{te}$, so that the friction force agrees with that given in [4] (see next Sec. 3 and Appendix). The 2nd order moment is the electron temperature, which is anisotropic in the parallel and perpendicular to $\boldsymbol{B}$ directions; $T_{e\parallel} = \langle m_e v_\parallel^2 \rangle$, and $T_{e\perp} = \langle m_e v_\perp^2/2 \rangle$. The effective isotropic temperature is given by $T_e = (T_{e\parallel} + 2T_{e\perp})/3$, and the anisotropy in the temperature, $\Delta_T = 2(T_{e\parallel} - T_{e\perp})/3T_e$, is generated by an AT distortion $\delta f_{\Delta T}$. Kinetic simulations for SOL-divertor plasmas by a particle modeling (PARASOL) showed a remarkable discrepancy between $T_{e\parallel}$ and $T_{e\perp}$ ($\Delta_T \sim -0.5$) for low collisionality [5, 6]. The 4th order moments can be expressed with the square of the 2nd order moment including Δ_T effect as,

$$\langle m_e v_\parallel^4/2 \rangle = (3/2 + 3\Delta_T)T_e^2/m_e \approx (3/2)T_{e\parallel}^2/m_e, \quad (4)$$

$$\langle m_e v_\parallel^2 v_\perp^2/2 \rangle = (1 + \Delta_T/2)T_e^2/m_e \approx T_{e\parallel}T_{e\perp}/m_e, \quad (5)$$

which are necessary for the transport equation of the heat flux q_e [7].

The conductive heat flux q_e is generated by the asymmetry in the velocity spread, i.e., skewness. One of major sources of the velocity-spread asymmetry (VSA) is the temperature gradient, $\nabla_\parallel T_e$. At first, we assume a skewed Maxwellian distribution for an isotropic VSA, $f_{SkM} \sim \exp\{-m_e v^2/2(1 + \varepsilon z v/v_t)T_e\}$. The isotropic VSA is in proportion to zv (ε is a small parameter). The "heat-flux (HF) distortion", δf_q, is derived by expanding f_{SkM}; $\delta f_q \sim z(v^3/5v_{te}^3 - v/v_{te})f_M$. It is noted that the HF distortion is set not to generate the mean flow; $\langle v_\parallel \rangle = (1/n_e)\int d\boldsymbol{v}(v_\parallel \delta f_q) = 0$. A function, $(v^3/5v_{te}^3 - v/v_{te})$, makes $\langle v_\parallel \rangle = 0$ by $\int dv$ integration. The total heat flux can be divided into parallel component and perpendicular component, $q_e = q_{e\parallel} + q_{e\perp}$; $q_e = \int d\boldsymbol{v}(m_e v_\parallel v^2/2)\delta f_q$ and $q_{e\parallel} = \int d\boldsymbol{v}(m_e v_\parallel^3/2)\delta f_q$. For the case of isotropic VSA, their percentages are fixed as $q_{e\parallel}/q_e = 3/5$ and $q_{e\perp}/q_e = 2/5$.

Next, we consider the anisotropic VSA; $f_{SkB} \sim \exp\{-m_e v_\parallel^2/2(1 + \varepsilon_\parallel z v/v_{te})T_e\} \times \exp\{-m_e v_\perp^2/2(1 + \varepsilon_\perp z v/v_{te}) T_e\}$. The above percentages are varied by the anisotropic VSA as $q_{e\parallel}/q_e = 3/5 + r_q$ and $q_{e\perp}/q_e = 2/5 - r_q$, where $r_q = (2q_{e\parallel} - 3q_{e\perp})/5q_e$ is the "HF anisotropy" in the range of $-3/5 < r_q < 2/5$. An HF distortion function consists of two parts, $\delta f_q = \delta f_{q1} + \delta f_{q2}$, total flux part δf_{q1} and anisotropic flux part δf_{q2};

$$\delta f_{q1} = (q_e/q_e^{FS})z(v^3/5v_{te}^3 - v/v_{te})G_{q1}f_M, \quad (6)$$

$$\delta f_{q2} = r_q(q_e/q_e^{FS})(5z^3 - 3z)(v^3/6v_{te}^3)G_{q2}f_M. \quad (7)$$

The magnitude of q_e is based on the free-streaming-energy flux, $q_e^{FS} = v_{te}n_e T_e$. A skewed Maxwellian distribution is further deformed by collisions. Similarly to the MF dissipation for reasonable convenience, we put functions in Eqs. (6) and (7), $\mathrm{G}_{q1,2} = \mathrm{G}_{q1,2}(\xi, Z_{eff})$, so that the heat conductivity and the thermal force agree with those given in [4]. As noted above, the HF distortion does not create the mean flow. A function, $(5z^3 - 3z)$, for δf_{q2} promises $\langle v_\parallel \rangle = 0$ by $\int dz$ integration.

On the other hand, the current-carrying heat flux q^J comes from δf_U, and its components are approximately given by $q^J{}_\parallel = 3Un_eT_e/2$ and $q^J{}_\perp = Un_eT_e$, respectively. The percentages are $q^J{}_\parallel/q^J = 3/5$ and $q^J{}_\perp/q^J = 2/5$, which coincide with those of conductive heat flux for the isotropic condition, $q_{e\parallel}/q_e = 3/5$ and $q_{e\perp}/q_e = 2/5$.

2.2 Simplified Fokker-Planck equation

The above distortions are dissipated by Coulomb collisions. Non-thermal-equilibrium quantities, U, Δ_T, q_e and r_q, are relaxed in the fluid moment equations. In order to evaluate quantitatively the relaxation processes, we apply a Fokker-Planck equation for electron distribution function f_e in a one-dimensional system along $\boldsymbol{B}$.

$$\partial_t f_e + v_\parallel \partial_s f_e - (eE_\parallel/m_e)\partial_{v\parallel} f_e = C(f_e) + S, \quad (8)$$

where t is the time and s is a coordinate parallel to $\boldsymbol{B}$. Partial differential operators are $\partial_t \equiv \partial/\partial t$, $\partial_s \equiv \partial/\partial s$ and $\partial_{v\parallel} = \partial/\partial v_\parallel$, respectively. Variables are the elementary charge e, the electric field $E_\parallel$ parallel to $\boldsymbol{B}$ and the source/sink term S for particle, momentum, energy and heat flux.

The simplified collision term consists of two terms, $C(f_e) = C^{e/i}(f_e) + C^{e/e}(f_e)$,

$$C^{e/i}(f_e) = (Z_{eff}/2\tau_{e1})\partial_z[(1 - z^2)\partial_z \delta f_e], \quad (9)$$

$$C^{e/e}(f_e) = (1/c_z\langle \tau_{e1} \rangle)\partial_z[(1 - z^2)\partial_z \delta f_e], \quad (10)$$

where Eq. (9) corresponds to the pitch-angle scattering (PAS) with ions, and Eq. (10) corresponds to the PAS within electrons. The energy diffusion term is ignored. A basic collision time depending on υ^3, $\tau_{e1} = 4\pi\varepsilon_0{}^2 m_e{}^2\upsilon^3/e^4 n_e \Lambda$ (ε_0: permittivity of free space) [8], is adopted for e/i collision, while an averaged collision time, $\langle\tau_{e1}\rangle \equiv (2\upsilon_t{}^2/\upsilon^2)^{3/2}\tau_{e1} = (4/3\pi^{1/2})Z_{eff}\tau_{e0}$ defined at $m_e\upsilon^2/2 = T_e$, is used for e/e collision, for simplicity.

The AT distortion $\delta f_{\Delta T}$ is dissipated by the PAS. The self-relaxation time for the anisotropic temperature is defined as $dT_{e\|}/dt = -2(T_{e\|} - T_{e\perp})/\tau_{rel}{}^{e/e}$ with $\tau_{rel}{}^{e/e} = 15(2\pi)^{1/2}\langle\tau_{e1}\rangle/8 = 5Z_{eff}\tau_{e0}/2^{1/2}$ [8]. The RHS of the above equation is given by the 2nd moment of the e/e collision term; $\int d\boldsymbol{\upsilon}(m_e\upsilon^2 z^2)C^{e/e}(\delta f_{\Delta T}) = -6n_e\Delta_T T_e/c_z\langle\tau_{e1}\rangle = -4n_e(T_{e\|} - T_{e\perp})/c_z\langle\tau_{e1}\rangle$. The coefficient is then determined as $c_z = 15(2\pi)^{1/2}/4$.

3. Relaxation Terms for Fluid Moment Equations

3.1 Momentum balance equation

The electron fluid momentum equation is simplified to a generalized Ohm's law by neglecting the inertia and source terms;

$$e n_e E_\| = -\partial_s(n_e T_{e\|}) + F_{fr} + F_{th}, \tag{11}$$

where $-\partial_s(n_e T_{e\|})$ is the pressure-gradient force. The friction force F_{fr} parallel to $\boldsymbol{B}$ comes from the e/i collision term as

$$F_{fr} \equiv \int d\boldsymbol{\upsilon}(m_e\upsilon_\|)C^{e/i}(\delta f_U) = -\alpha_0 n_e m_e U/\tau_{e0}. \tag{12}$$

The "friction-force coefficient" is given by α_0. Its value is $\alpha_0 = 0.51$ for $Z_{eff} = 1$, and is decreased gradually with Z_{eff}. The nonlinear dependence on Z_{eff} comes from the fact that the relaxation process is affected by both the e/e collision, $C^{e/e} \propto Z_{eff}{}^0/\upsilon^0$, and the e/i collision, $C^{e/i} \propto Z_{eff}/\upsilon^3$ (see Sec. 2.2). The value of α_0 was shown in TABLE 2 of [4] by "Braginskii". Hereafter we denote this listed value as $\alpha_0{}^{BR}$. For the availability on the continuous Z_{eff} dependence of α_0, we devise a simple and convenient expression,

$$\alpha_0{}^{fit} = (0.77 + 0.35Z_{eff})/(1 + 1.2Z_{eff}). \tag{13}$$

Table 1 compares $\alpha_0{}^{fit}$ and $\alpha_0{}^{BR}$ for several Z_{eff} values (Z_{eff} = 1, 2, 3, 4 and ∞). There are also listed "*exact*" values at Z_{eff} = 1 and ∞, which were written nearby TABLE 2 of [4]. One sees satisfactory agreement between them. How to construct this fitting function is presented in Appendix.

The thermal force F_{th} is generated by the relaxation of HF distortion δf_q due to e/i PAS depending on $1/\upsilon^3$. This force is linearly proportional to q_e and inversely proportional to τ_{e0},

$$F_{th} \equiv \int d\boldsymbol{\upsilon}(m_e\upsilon_\|)C^{e/i}(\delta f_q) = A_{th} q_e/\upsilon_{te}{}^2\tau_{e0}, \tag{14}$$

where we call A_{th} the "basic thermal-force coefficient". It is noted that the HF anisotropy r_q does not appear in Eq. (14) (see Appendix). This formula is explicitly different from the conventional one [4],

$$F_{th,\nabla_\| T} = -\beta_0 n_e \nabla_\| T_e, \tag{15}$$

Table 1. Z_{eff} dependences of α_0, β_0, γ_0, and $A_{q*} = 5(1-\beta_0)/2\gamma_0$.

Z_{eff}	1	2	3	4	∞
$\alpha_0{}^{BR}$	0.5129	0.441	0.397	0.375	0.2949
exact	*0.5063*				*3π/32*
$\alpha_0{}^{fit}$	0.509	0.432	0.396	0.374	0.292
$\beta_0{}^{BR}$	0.7110	0.905	1.016	1.090	1.521
exact	*0.7033*				*3/2*
$\beta_0{}^{fit}$	0.7083	0.894	1.009	1.087	1.500
$\gamma_0{}^{BR}$	3.162	4.890	6.064	6.920	12.47
exact	*3.203*				*128/3π*
$\gamma_0{}^{fit}$	3.200	4.900	6.114	7.025	13.40
$A_{q*}{}^{BR}$	0.2285	0.0485	−0.007	−0.033	−0.104
exact	*0.2316*				*−0.092*
$A_{q*}{}^{fit}$	0.233	0.067	0.011	−0.017	−0.100

which is linearly proportional to $\nabla_\| T_e$ and independent of τ_{e0}. Though Eqs. (14) and (15) seem fully different each other, they are equivalent when the conductive heat flux is given by the conventional formula so called "Spitzer-Hahm (SH)" heat flux,

$$q_e = q_e{}^{SH} \equiv -\gamma_0 n_e \upsilon_{te}{}^2 \tau_{e0}\nabla_\| T_e. \tag{16}$$

Equation (14) is rewritten as $F_{th} = -A_{th}\gamma_0 n_e \nabla_\| T_e$, and an essential relation is found;

$$A_{th} = \beta_0/\gamma_0. \tag{17}$$

In a rare-collision plasma, however, $q_e{}^{SH}$ can become much larger than $q_e{}^{FS}$ unphysically, and some kinetic effects (or flux limiting effects) on q_e have been added to the fluid simulation modeling, such as a harmonic average model, $1/q_e = 1/q_e{}^{SH} + 1/\alpha_e q_e{}^{FS}$ (α_e: electron heat-flux-limiting factor) [6, 7, 9]. Therefore, it is necessary to use Eq. (14) directly instead of the conventional formula, Eq. (15).

Values of β_0 and γ_0 were listed in TABLE 2 of [4]. Dependences of β_0 and γ_0 on Z_{eff} are approximated by the expressions similar to Eq. (13) for α_0,

$$\beta_0{}^{fit} = (0.36 + 0.66Z_{eff})/(1 + 0.44Z_{eff}), \tag{18}$$

$$\gamma_0{}^{fit} = (0.65 + 3.35Z_{eff})/(1 + 0.25Z_{eff}). \tag{19}$$

Figure 1, as well as Table 1, shows that approximate formulae of $\alpha_0{}^{fit}$, $\beta_0{}^{fit}$ and $\gamma_0{}^{fit}$ fit very well those values given in [4]. The approximation of A_{th} is simply given by $A_{th}{}^{fit} = \beta_0{}^{fit}/\gamma_0{}^{fit}$.

3.2 Transport equations for temperature

Basing on the above simplified Boltzmann transport equation, the 2nd order moment equation of an electron fluid for the effective isotropic temperature T_e is obtained as

$$(3/2)n_e\partial_t T_e + \partial_s q_e = Q, \tag{20}$$

where Q is the total heating/cooling power density. The current-carrying heat transport is neglected, for simplicity. When taking into account the anisotropic temperatures, two transport equations for $T_{e\|}$ and $T_{e\perp}$ are treated separately;

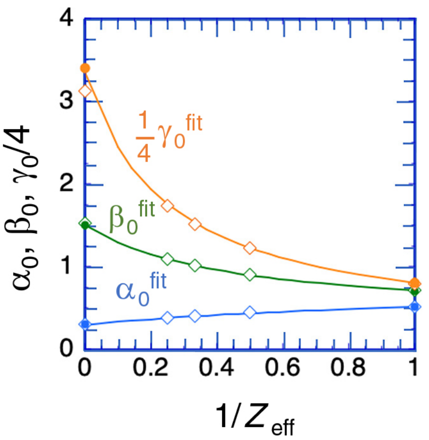

Fig. 1. Fitting curves of relaxation coefficients, $\alpha_0{}^{\mathrm{fit}}$, $\beta_0{}^{\mathrm{fit}}$ and $\gamma_0{}^{\mathrm{fit}}$. Open diamonds are those listed in TABLE 2 of [4], and closed circles are those written as "exact" values in [4].

$$(1/2)n_{\mathrm{e}}\partial_t T_{\mathrm{e}\|} + \partial_s q_{\mathrm{e}\|} = Q_\| - \mathrm{A}_{\Delta T} n_{\mathrm{e}}(T_{\mathrm{e}\|} - T_{\mathrm{e}\perp})/\tau_{\mathrm{e}0}, \tag{21}$$

$$n_{\mathrm{e}}\partial_t T_{\mathrm{e}\perp} + \partial_s q_{\mathrm{e}\perp} = Q_\perp + \mathrm{A}_{\Delta T} n_{\mathrm{e}}(T_{\mathrm{e}\|} - T_{\mathrm{e}\perp})/\tau_{\mathrm{e}0}, \tag{22}$$

where $Q_\|$ heats up $T_{\mathrm{e}\|}$ and $Q_\perp$ heats up $T_{\mathrm{e}\perp}$, separately. For the case of isotropic heating/cooling, Q is divided into the parallel component $Q_\| = Q/3$ and the perpendicular component $Q_\perp = 2Q/3$, respectively. The "AT relaxation coefficient" $\mathrm{A}_{\Delta T}$ is given by

$$\mathrm{A}_{\Delta T} = 2/5 + 2^{1/2}/5Z_{\mathrm{eff}}, \tag{23}$$

where the 2nd part is equal to $\tau_{\mathrm{e}0}/\tau_{\mathrm{rel}}{}^{\mathrm{e/e}}$. These set of equations for $T_{\mathrm{e}\|}$ and $T_{\mathrm{e}\perp}$ are rewritten to a transport equation of $\Delta_T T_{\mathrm{e}}$;

$$(3/2)n_{\mathrm{e}}\partial_t(\Delta_T T_{\mathrm{e}}) + \partial_s(2q_{\mathrm{e}\|} - q_{\mathrm{e}\perp}) = (2Q_\| - Q_\perp) - 9\mathrm{A}_{\Delta T} n_{\mathrm{e}}\Delta_T T_{\mathrm{e}}/2\tau_{\mathrm{e}0}. \tag{24}$$

Even for the isotropic heating/cooling case ($2Q_\| = Q_\perp$), or no heating/cooling case ($Q_\| = Q_\perp = 0$), the temperature anisotropy $\Delta_T T_{\mathrm{e}}$ can be driven by the 2nd term of LHS; unbalance in the heat flux, $2q_{\mathrm{e}\|} - q_{\mathrm{e}\perp} = (4/5 + 3r_q)q_{\mathrm{e}}$. This unbalance exists naturally for the isotropic heat flux, $r_q = 0$. Accordingly, we have to solve the transport of heat fluxes, $q_{\mathrm{e}\|}$ and $q_{\mathrm{e}\perp}$, simultaneously with Eq. (24).

3.3 Transport equations for heat flux

The 3rd order moment equation for the total conductive heat flux q_{e} is obtained as

$$\partial_t q_{\mathrm{e}} + \partial_s(n_{\mathrm{e}}M_4) + (5/2 + \Delta_T)eE_\| n_{\mathrm{e}}v_{\mathrm{te}}{}^2 = -\mathrm{A}_{q^*}q_{\mathrm{e}}/\tau_{\mathrm{e}0}, \tag{25}$$

where $M_4 \equiv \langle m_{\mathrm{e}}v_\|{}^2 v^2/2\rangle$ is a 4th order moment explained below in detail. We neglect the current-carrying heat transport and the source/sink term for q_{e}, for simplicity.

The collision term $\int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\| v^2/2)C(\delta f_q) = \int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\| v^2/2)$ $C(\delta f_{q1})$ is expressed by $-\mathrm{A}_{q^*}q_{\mathrm{e}}/\tau_{\mathrm{e}0}$ in the RHS. It is interestingly noted that the "HF rate coefficient" A_{q^*} in Eq. (25) is not always the damping one ($\mathrm{A}_{q^*} > 0$) but sometimes the growing one ($\mathrm{A}_{q^*} < 0$) due to the v^3 dependence of the e/i collision time $\tau_{\mathrm{e}1}$. Considering $q_{\mathrm{e}} > 0$, δf_q consists of "forward" higher-energy part with $v_{\|+} > 0$ and "backward" lower-energy part with $v_{\|-} < 0$. The larger scattering of low-energy electrons by ions reduces faster the backward heat flux, while smaller scattering of high-energy electrons reduces slower the forward heat flux. As a result, an increase in the net forward heat flux can arise. When $Z_{\mathrm{eff}} \gg 1$, the PAS by e/i collision with $\tau^{\mathrm{e/i}} \propto v^3$ is dominant. We evaluate $q_{\mathrm{e}} \sim (n_+v_{\|+}T_+ + n_-v_{\|-}T_-)$, and obtain $q_{\mathrm{e}} \sim n_{\mathrm{e}}v_{\mathrm{te}}(T_+ - T_-)$ under a condition of $\langle v_\| \rangle = 0$. Then we can approximate the temporal change of q_{e} for $Z_{\mathrm{eff}} \gg 1$ as, $(1/n_{\mathrm{e}}v_{\mathrm{te}})\mathrm{d}q_{\mathrm{e}}/\mathrm{d}t \sim \mathrm{d}/\mathrm{d}t$ $(T_+ - T_-) \sim -(T_+/\tau_+ - T_-/\tau_-) \sim -(T_{\mathrm{e}}{}^{3/2}/\tau^{\mathrm{e/i}})(1/T_+{}^{1/2} - 1/T_-{}^{1/2}) \sim (T_+ - T_-)/\tau^{\mathrm{e/i}} \sim +(1/n_{\mathrm{e}}v_{\mathrm{te}})(q_{\mathrm{e}}/\tau^{\mathrm{e/i}})$. When Z_{eff} approaches to unity and the PAS by e/e collision is effective, A_{q^*} becomes positive. The above collisional growth of q_{e} was not clearly mentioned in the previous works, *e.g.*, [7]. It is noted again a well-known fact that the thermal force is generated due to v^3 dependence of $\tau^{\mathrm{e/i}}$; $F_{\mathrm{th}} = \mathrm{d}/\mathrm{d}t(m_{\mathrm{e}}n_{\mathrm{e}}\langle v_\|\rangle) = \mathrm{A}_{\mathrm{th}}q_{\mathrm{e}}/v_{\mathrm{te}}{}^2\tau^{\mathrm{e/i}}$ (see Eq. (14)).

The 4th order moment, $n_{\mathrm{e}}M_4 \equiv n_{\mathrm{e}}\langle m_{\mathrm{e}}v_\|{}^2v^2/2\rangle$, in the LHS of Eq. (25) is given from Eqs. (4) and (5) as; $n_{\mathrm{e}}M_4 = (5/2 + 7\Delta_T/2)n_{\mathrm{e}}T_{\mathrm{e}}v_{\mathrm{te}}{}^2 = (5/2)n_{\mathrm{e}}T_{\mathrm{e}\|}v_{\mathrm{te}}{}^2 + \Delta_T n_{\mathrm{e}}T_{\mathrm{e}}v_{\mathrm{te}}{}^2$. Its gradient is expanded as; $\partial_s(n_{\mathrm{e}}M_4) \approx (5/2 + \Delta_T)v_{\mathrm{te}}{}^2\partial_s(n_{\mathrm{e}}T_{\mathrm{e}\|}) + (5/2)(1 + \Delta_T)n_{\mathrm{e}}v_{\mathrm{te}}{}^2\partial_s T_{\mathrm{e}} + (n_{\mathrm{e}}v_{\mathrm{te}}{}^2)\partial_s(\Delta_T T_{\mathrm{e}})$. Considering the force balance of the generalized Ohm's law, Eq. (11), the above $\partial_s(n_{\mathrm{e}}M_4)$ is rewritten as follows; $\partial_s(n_{\mathrm{e}}M_4) + (5/2 + \Delta_T)eE_\| n_{\mathrm{e}}v_{\mathrm{te}}{}^2 \approx (5/2 + \Delta_T)\mathrm{A}_{\mathrm{th}}q_{\mathrm{e}}/\tau_{\mathrm{e}0} + (5/2)(1 + \Delta_T)n_{\mathrm{e}}v_{\mathrm{te}}{}^2\partial_s T_{\mathrm{e}} + (n_{\mathrm{e}}v_{\mathrm{te}}{}^2)\partial_s(\Delta_T T_{\mathrm{e}})$.

Taking into account this thermal-force effect on $\partial_s(n_{\mathrm{e}}M_4)$, Eq. (25) is reduced to

$$\partial_t q_{\mathrm{e}} + (5/2)(1 + \Delta_T)n_{\mathrm{e}}v_{\mathrm{te}}{}^2\partial_s T_{\mathrm{e}} + n_{\mathrm{e}}v_{\mathrm{te}}{}^2\partial_s(\Delta_T T_{\mathrm{e}}) = -\mathrm{A}_q q_{\mathrm{e}}/\tau_{\mathrm{e}0}. \tag{26}$$

A new relaxation coefficient, $\mathrm{A}_q \equiv \mathrm{A}_{q^*} + (5/2 + \Delta_T)\mathrm{A}_{\mathrm{th}}$, is called "HF relaxation coefficient". This collision term is always the damping type ($\mathrm{A}_q > 0$). When we force $\Delta_T = 0$, we obtain a steady-state relation similar to the SH heat flux; $q_{\mathrm{e}} = -(5/2\mathrm{A}_q)n_{\mathrm{e}}v_{\mathrm{te}}{}^2\tau_{\mathrm{e}0}\partial_s T_{\mathrm{e}}$. By comparing with Eq. (16), we immediately find $\mathrm{A}_q|_{\Delta_T=0} = 5/2\gamma_0$. The HF rate coefficient is then determined as $\mathrm{A}_{q^*} = 5/2\gamma_0 - 5\mathrm{A}_{\mathrm{th}}/2 = 5(1 - \beta_0)/2\gamma_0$. Although A_{q^*} is approximated by $\mathrm{A}_{q^*} \approx 5(1 - \beta_0{}^{\mathrm{fit}})/2\gamma_0{}^{\mathrm{fit}}$, we apply a much simpler expression, a linear function of $1/Z_{\mathrm{eff}}$,

$$\mathrm{A}_{q^*}{}^{\mathrm{fit}} = (1 - 0.3Z_{\mathrm{eff}})/3Z_{\mathrm{eff}}. \tag{27}$$

This Z_{eff} dependence agrees fairly well with that of $\mathrm{A}_{q^*} \equiv 5(1 - \beta_0)/2\gamma_0$ (see Table 1). The HF relaxation coefficient A_q is approximated by applying the fitting expressions, $\gamma_0{}^{\mathrm{fit}}$ and $\mathrm{A}_{\mathrm{th}}{}^{\mathrm{fit}} = \beta_0{}^{\mathrm{fit}}/\gamma_0{}^{\mathrm{fit}}$,

$$\mathrm{A}_q{}^{\mathrm{fit}} = 5/2\gamma_0{}^{\mathrm{fit}} + \Delta_T\mathrm{A}_{\mathrm{th}}{}^{\mathrm{fit}}. \tag{28}$$

In order to calculate $\partial_s(\Delta_T T_{\mathrm{e}})$ in Eq. (26), the transport of anisotropic heat fluxes, $q_{\mathrm{e}\|}$ and $q_{\mathrm{e}\perp}$, have to be solved simultaneously. The transport equations for $q_{\mathrm{e}\|}$ and $q_{\mathrm{e}\perp}$ are given separately as follows,

$$\partial_t q_{e\parallel} + (3/2)(1+\Delta_T)n_e v_{te}^2 \partial_s T_e + (3/2)n_e v_{te}^2 \partial_s(\Delta_T T_e) = -3A_q q_e/5\tau_{e0} - A_r(2q_{e\parallel} - 3q_{e\perp})/5\tau_{e0}, \quad (29)$$

$$\partial_t q_{e\perp} + (1+\Delta_T)n_e v_{te}^2 \partial_s T_e - (1/2)n_e v_{te}^2 \partial_s(\Delta_T T_e) = -2A_q q_e/5\tau_{e0} + A_r(2q_{e\parallel} - 3q_{e\perp})/5\tau_{e0}. \quad (30)$$

The 1st collision terms in RHS of Eqs. (29) and (30) concern the HF distortion δf_{q1}; $\int d\boldsymbol{v}(m_e v_\parallel{}^3/2)C(\delta f_{q1}) = -3A_{q*}q_e/5\tau_{e0}$ with $A_q \equiv A_{q*} + (5/2+\Delta_T)A_{th}$. On the other hand, the 2nd collision terms concern the anisotropic part δf_{q2}; $\int d\boldsymbol{v}(m_e v_\parallel{}^3/2)$ $C(\delta f_{q2}) = -A_r r_q q_e/\tau_{e0}$ with $r_q \equiv (2q_{e\parallel} - 3q_{e\perp})/5q_e$. It is interesting that the relaxation processes are fully coupled between $q_{e\parallel}$ and $q_{e\perp}$ due to the PAS, $(1/\tau)\partial_z[(1-z^2)\partial_z \delta f_q]$. In the previous works, *e.g.* [7], this coupling was not taken into account. Because we cannot find suitable reference values for the curve fitting on the “HF anisotropy relaxation coefficient” A_r, we create an approximate formula similar to Eq. (27) for $A_{q*}{}^{fit}$.

$$A_r{}^{cre} = (2 + 0.8Z_{eff})/Z_{eff}. \quad (31)$$

To create this linear function of $1/Z_{eff}$, we apply two values of $A_r = -(\tau_{e0}/r_q q_e)\int d\boldsymbol{v}(m_e v_\parallel{}^3/2)C(\delta f_{q2})$ with $G_{q2} \propto \xi^3$ for $Z_{eff} \gg 1$ and $G_{q2} \propto \xi^2$ for $Z_{eff} \approx 2.7$ (see Appendix).

Figure 2 shows fitting curves (dashed lines) of the relaxation coefficients, $\alpha_0{}^{fit}$, $A_{th}{}^{fit}$ and $A_{q*}{}^{fit}$ dependent on $1/Z_{eff}$. Open circles are obtained by setting $G \propto \xi^3$ for $1/Z_{eff} = 0$ and $G \propto \xi^2$ for $1/Z_{eff} = 1/2.7$, which agree well with own fitting curve. A red solid line for $A_r{}^{cre}$ passes through two points at $1/Z_{eff} = 0$ and 1/2.7 (red closed circles). The slope 2 of $A_r{}^{cre}$ is set just six times larger than 1/3 of $A_{q*}{}^{fit}$, because $C^{e/e}(\delta f_{q1}) = -(2^{1/2}/5Z_{eff}\tau_{e0})\delta f_{q1}$ and $C^{e/e}(\delta f_{q2}) = -6 \times (2^{1/2}/5Z_{eff}\tau_{e0})\delta f_{q2}$.

It is noted that a sum of Eqs. (29) and (30) becomes the total heat flux transport equation of Eq. (26). Further, a subtraction, (2/5) × Eq. (29) − (3/5) × Eq. (30), becomes the transport equation for the anisotropic heat flux;

$$\partial_t(r_q q_e) + (9/10)(n_e v_{te}^2)\partial_s(\Delta_T T_e) = -A_r r_q q_e/\tau_{e0}. \quad (32)$$

This equation for $r_q q_e$ includes $\partial_s(\Delta_T T_e)$ term but does not include $\partial_s T_e$ term, while Eq. (26) for q_e includes both $\partial_s T_e$ and $\partial_s(\Delta_T T_e)$ terms.

We complete the five transport equations of electron fluid moments, Eq. (11) of the force balance, Eq. (20) for T_e, Eq. (24) for $\Delta_T T_e$, Eq. (26) for q_e and Eq. (32) for $r_q q_e$. Anisotropies in T_e and q_e are taken into account. Another set of five equations is equivalently available; Eqs. (11) and (21) for $T_{e\parallel}$, Eq. (22) for $T_{e\perp}$, Eq. (29) for $q_{e\parallel}$ and Eq. (30) for $q_{e\perp}$. The electron continuity equation is not always necessary to be solved, but the ion continuity equation is solved under the condition of charge neutrality.

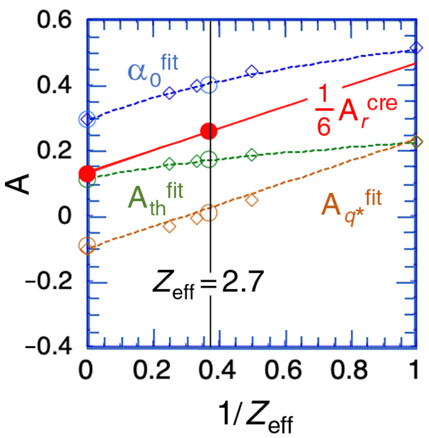

Fig. 2. Fitting curves of relaxation coefficients, $\alpha_0{}^{fit}$, $A_{th}{}^{fit} = \beta_0{}^{fit}/\gamma_0{}^{fit}$ and $A_{q*}{}^{fit}$. Diamonds are those given in [4]. Circles are analytical ones. HF anisotropy relaxation coefficient $A_r{}^{cre}$ is created so that the line passes through two points (red circles at $1/Z_{eff} = 0$ and 1/2.7).

4. Summary and Discussion

In this paper, we study the collisional relaxation processes of electron fluid moments in open magnetic field systems, such as SOL-divertor plasmas, by using a simplified Fokker-Planck equation. The thermal force is given not by the temperature gradient, $F_{th} = -\beta_0 n_e \nabla_\parallel T_e$, but by the heat flux damping, $F_{th} = A_{th} q_e/v_{te}{}^2\tau_{e0}$. We devise approximate formulae representing the Z_{eff} dependence of relaxation coefficients, “friction-force coefficient” α_0, “thermal-force coefficient” β_0 and “thermal-conductivity coefficient” γ_0;

$$\alpha_0{}^{fit} = (0.77 + 0.35Z_{eff})/(1 + 1.2Z_{eff}),$$

$$\beta_0{}^{fit} = (0.36 + 0.66Z_{eff})/(1 + 0.44Z_{eff}),$$

$$\gamma_0{}^{fit} = (0.65 + 3.35Z_{eff})/(1 + 0.25Z_{eff}).$$

These expressions can be easily utilized in general divertor simulation codes using conventional plasma fluid equations.

Anisotropies in the temperature, $\Delta_T \equiv 2(T_{e\parallel} - T_{e\perp})/3T_e$, and in the conductive heat flux, $r_q = (2q_{e\parallel} - 3q_{e\perp})/5q_e$, are taken into account. A relaxation between $q_{e\parallel}$ and $q_{e\perp}$ is definitely described in the transport equation, $\partial_t(2q_{e\parallel} - 3q_{e\perp}) \sim -A_r(2q_{e\parallel} - 3q_{e\perp})/\tau_{e0}$. The “heat-flux anisotropy relaxation coefficient” A_r is derived and its approximation formula is created, $A_r{}^{cre} = (2 + 0.8Z_{eff})/Z_{eff}$. We complete a set of five electron fluid moment equations; a force balance equation, and four transport equations of T_e, $\Delta_T T_e$, q_e and $r_q q_e$. We will study, in the near future, the electron heat transport along $\boldsymbol{B}$ in open magnetic field systems by utilizing these electron fluid moment equations. The conductive heat flux is not simply be expressed as $q_e = q_e{}^{SH}$ nor $1/q_e = 1/q_e{}^{SH} + 1/\alpha_e q_e{}^{FS}$. The flux-limiting effect in a rare-collision plasma, which has been sometimes called kinetic effect, can be described to some extent by solving directly the transport equation of q_e.

We have developed the anisotropic-ion-pressure (AIP) fluid model for the SOL-divertor plasma simulation [10, 11]. To simulate more accurately the plasma transport in a rare-collision plasma, we are going to extend the AIP model by adding the heat flux transport equations. The present methodology will be useful. For the ion transport in SOL plasmas, however, the Mach number and temperature anisotropy become remarkably large [5, 6, 10]. Resultantly the present functional forms of δf generate the unacceptably-wide negative f region. Alternative proper forms of δf_{i} should be found out to complete the ion fluid moment equations.

Acknowledgements

We acknowledge Dr. K. Ibano of Starlight Engine Ltd., Dr. Y. Homma of QST and Prof. M. Sakamoto of University of Tsukuba for their collaborations and encouragements on our research activities. This work is partly supported by the NIFS Collaboration Research Program (NIFS25KFFT001).

Appendix. Relaxation Coefficients

The collision term given by Eqs. (9) and (10) for an MF distortion, $\delta f_U = (zvU/v_{\mathrm{te}}^2)\mathrm{G}_U f_{\mathrm{M}}$, is written as $C(\delta f_U) = -2(Z_{\mathrm{eff}}/2\tau_{\mathrm{e1}} + 1/\mathrm{c}_z\langle\tau_{\mathrm{e1}}\rangle)\delta f_U$, because $\partial_z[(1-z^2)\partial_z(z)] = -2z$. A stationary Fokker-Planck equation, $-(eE_\parallel/m_{\mathrm{e}})\partial_{v\parallel} f_{\mathrm{M}} = C(\delta f_U)$, is then reduced to a relation, $eE_\parallel/m_{\mathrm{e}} = (2U/\langle\tau_{\mathrm{e1}}\rangle)\{Z_{\mathrm{eff}}(2v_{\mathrm{te}}^2/v^2)^{3/2} + 1/\mathrm{c}_z\}\mathrm{G}_U$. We find that the adjustment function has a form of $\mathrm{G}_U(\xi, Z_{\mathrm{eff}}) \sim 1/(2^{3/2}Z_{\mathrm{eff}}/\xi^3 + 1/\mathrm{c}_z)$, where $\xi \equiv v/v_{\mathrm{te}}$. The friction force is given by $F_{\mathrm{fr}} = \int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\parallel) C^{\mathrm{e/i}}(\delta f_U) = -\alpha_0 n_{\mathrm{e}} m_{\mathrm{e}} U/\tau_{\mathrm{e0}}$. In case of $Z_{\mathrm{eff}} \gg 1$, G_U becomes simply proportional to ξ^3 and $\delta f_U[\mathrm{G}^{(3)}] = \{(2\pi)^{1/2}U/32v_{\mathrm{te}}\} z\xi^4 f_{\mathrm{M}}$. Brackets $[\mathrm{G}^{(3)}]$ denote a quantity calculated by setting $\mathrm{G} \propto \xi^3$, and later $[\mathrm{G}^{(2)}]$ denote a quantity calculated by setting $\mathrm{G} \propto \xi^2$. One obtains analytically $F_{\mathrm{fr}}[\mathrm{G}^{(3)}] = -(3\pi/32) n_{\mathrm{e}} m_{\mathrm{e}} U/\tau_{\mathrm{e0}}$, i.e., $\alpha_0[\mathrm{G}^{(3)}] = 3\pi/32$. This was written nearby TABLE 2 of [4] as an exact value for $Z_{\mathrm{eff}} = \infty$.

The above G_U is not enough accurate for $Z_{\mathrm{eff}} \sim \mathrm{O}(1)$, because the present e/e collision term is a simplified model without taking account of the relative-velocity effect and the energy-relaxation effect. Therefore, we apply an approximate expression of the "friction-force coefficient" α_0 representing the Z_{eff} dependence. Considering the form, $\mathrm{G}_U \sim 1/\{1 + \mathrm{g}(\xi)Z_{\mathrm{eff}}\}$, we introduce a variable, $\mathrm{X}(\mathrm{h}, Z_{\mathrm{eff}}) = 1/(1 + \mathrm{h}Z_{\mathrm{eff}})$, and carry out a linear fitting, $\alpha_0^{\mathrm{fit}} = \mathrm{a} + \mathrm{b}\mathrm{X}(\mathrm{h}, Z_{\mathrm{eff}})$, against 5 points of α_0^{BR} values in TABLE 2 of [4]. We obtain 3 coefficients, h = 1.2, a $\approx$ 0.29 and b $\approx$ 0.48, i.e., $\alpha_0^{\mathrm{fit}} = (0.77 + 0.35Z_{\mathrm{eff}})/(1 + 1.2Z_{\mathrm{eff}})$. Approximate formulae for β_0 and γ_0 are also obtained by this linear fitting technique; $\beta_0^{\mathrm{fit}} = (0.36 + 0.66Z_{\mathrm{eff}})/(1 + 0.44Z_{\mathrm{eff}})$ and $\gamma_0^{\mathrm{fit}} = (0.65 + 3.35Z_{\mathrm{eff}})/(1 + 0.25Z_{\mathrm{eff}})$, respectively. To obtain values of a and b for γ_0^{fit}, an exact value of $\gamma_0 = 128/3\pi$ at $Z_{\mathrm{eff}} = \infty$ is regarded.

In accordance with the above analysis, we calculate the "basic thermal-force coefficient" A_{th}. The collision terms for δf_{q1} and δf_{q2} are written as $C(\delta f_{q1}) = -2(Z_{\mathrm{eff}}/2\tau_{\mathrm{e1}} + 1/\mathrm{c}_z\langle\tau_{\mathrm{e1}}\rangle)\delta f_{q1} \propto z$ and $C(\delta f_{q2}) = -12(Z_{\mathrm{eff}}/2\tau_{\mathrm{e1}} + 1/c_z\langle\tau_{\mathrm{e1}}\rangle)\delta f_{q2} \propto (5z^3 - 3z)$, respectively. Because an integration, $\int \mathrm{d}z(m_{\mathrm{e}}v_\parallel)C(\delta f_{q2})$, is always null, the HF anisotropy r_q does not appear in Eq. (14) for F_{th}. In case of $Z_{\mathrm{eff}} \gg 1$ and $\mathrm{G}_{q1} \propto \xi^3$, the HF distortion is given by $\delta f_{q1}[\mathrm{G}^{(3)}] = \{(2\pi)^{1/2}/32\} (q_{\mathrm{e}}/q_{\mathrm{e}}^{\mathrm{FS}})z(\xi^6/8 - \xi^4)f_{\mathrm{M}}$. The thermal force is analytically obtained as $F_{\mathrm{th}}[\mathrm{G}^{(3)}] = \int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\parallel)C(\delta f_{q1}) = (9\pi/256)q_{\mathrm{e}}/v_{\mathrm{te}}^2\tau_{\mathrm{e0}}$, i.e., $\mathrm{A}_{\mathrm{th}}[\mathrm{G}^{(3)}] = 9\pi/256 \approx 0.110$. This is smaller a little than $\beta_0^{\mathrm{BR}}/\gamma_0^{\mathrm{BR}} \approx 0.122$ based on TABLE 2 of [4], but just the same as one based on exact values, $\beta_0 = 3/2$ and $\gamma_0 = 128/3\pi$, wrote in [4]. The approximation of A_{th} is simply given by $\mathrm{A}_{\mathrm{th}}^{\mathrm{fit}} = \beta_0^{\mathrm{fit}}/\gamma_0^{\mathrm{fit}}$.

The "HF-rate coefficient" A_{q*} can be approximated as $\mathrm{A}_{q*} \approx 5(1 - \beta_0^{\mathrm{fit}})/2\gamma_0^{\mathrm{fit}}$, but we apply a much simpler expression, a linear function of $1/Z_{\mathrm{eff}}$, $\mathrm{A}_{q*}^{\mathrm{fit}} = (1 - 0.3Z_{\mathrm{eff}})/3Z_{\mathrm{eff}}$, by fair agreements. The analytical evaluation, $-\mathrm{A}_{q*}q_{\mathrm{e}}/\tau_{\mathrm{e0}} = \int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\parallel v^2/2)C(\delta f_{q1})$ for $Z_{\mathrm{eff}} \gg 1$ and $\mathrm{G}_{q1} \propto \xi^3$, gives $\mathrm{A}_{q*}[\mathrm{G}^{(3)}] = -(15\pi/512) \approx -0.092$, which agrees with one based on exact values wrote in [4].

Since there was no suitable reference value for the curve fitting on the "HF anisotropy relaxation coefficient" A_r, we create an approximate formula similar to $\mathrm{A}_{q*}^{\mathrm{fit}}$. To create a linear function of $1/Z_{\mathrm{eff}}$, two points of $\mathrm{A}_r = -(\tau_{\mathrm{e0}}/r_q q_{\mathrm{e}}) \int \mathrm{d}\boldsymbol{v}(m_{\mathrm{e}}v_\parallel^3/2)C(\delta f_{q2})$ are necessary. One point is at $1/Z_{\mathrm{eff}} = 0$, which is calculated by setting $\mathrm{G}_{q2} \propto \xi^3$ and $\delta f_{q2}[\mathrm{G}^{(3)}] = \{7(2\pi)^{1/2}/3072\}r_q(q_{\mathrm{e}}/q_{\mathrm{e}}^{\mathrm{FS}})(5z^3 - 3z)\xi^6 f_{\mathrm{M}}$. The result is $\mathrm{A}_r[\mathrm{G}^{(3)}] = 63\pi/256 \approx 0.77$. Another point is calculated by setting artificially $\mathrm{G}_{q2} \propto \xi^2$ and $\delta f_{q2}[\mathrm{G}^{(2)}] = (1/54)r_q(q_{\mathrm{e}}/q_{\mathrm{e}}^{\mathrm{FS}})(5z^3 - 3z) \xi^5 f_{\mathrm{M}}$. The result is $\mathrm{A}_r[\mathrm{G}^{(2)}] = 32/35 + 6(2^{1/2}/5Z_{\mathrm{eff}})$. In order to select a proper Z_{eff} value corresponding to $\mathrm{G}_{q2} \propto \xi^2$, we calculate former relaxation coefficients for $\mathrm{G} \propto \xi^2$; $\alpha_0[\mathrm{G}^{(2)}] = 2/5$, $\mathrm{A}_{\mathrm{th}}[\mathrm{G}^{(2)}] = 6/35$, and $\mathrm{A}_{q*}[\mathrm{G}^{(2)}] = 4/35 - 2^{1/2}/5Z_{\mathrm{eff}}$. The MF distortion is $\delta f_U[\mathrm{G}^{(2)}] = (U/5v_{\mathrm{te}})z\xi^3 f_{\mathrm{M}}$. Each value coincides with each fitting expression at a certain Z_{eff} value; $Z_{\mathrm{eff}} \approx 2.9$ for α_0, $Z_{\mathrm{eff}} \approx 2.7$ for A_{th}, and $Z_{\mathrm{eff}} \approx 2.7$ for A_{q*}. By putting $\mathrm{A}_r[\mathrm{G}^{(2)}] \approx 1.54$ at $Z_{\mathrm{eff}} \approx 2.7$, we obtain the slope for $1/Z_{\mathrm{eff}}$ is about $2.7(1.54 - 0.77) \approx 2$, which is six times larger than that of $\mathrm{A}_{q*}^{\mathrm{fit}}$. This magnification is quite reasonable, because e/e collision terms, Eq. (10), are related as $C^{\mathrm{e/e}}(\delta f_{q1}) = -(2^{1/2}/5Z_{\mathrm{eff}}\tau_{\mathrm{e0}})\delta f_{q1}$ and $C^{\mathrm{e/e}}(\delta f_{q2}) = -6 \times (2^{1/2}/5Z_{\mathrm{eff}}\tau_{e0})\delta f_{q2}$. Finally, we chose a very simple form of $\mathrm{A}_r^{\mathrm{cre}}$ with an intercept 0.8; $\mathrm{A}_r^{\mathrm{cre}} = (2 + 0.8Z_{\mathrm{eff}})/Z_{\mathrm{eff}}$ (see Fig. 2 in Sec. 3.3).

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