

Periodization of Chaotic Oscillations by Modulation of Discharge Voltage: System Response to Changes in Square-Wave Duty Cycle

Hiroto MIYOSHI^{1)*}, Takao FUKUYAMA^{2)**}, Kenichiro TERASAKA³⁾, Yusuke KOSUGA^{4,5)}

¹⁾ *Interdisciplinary Graduate School of Engineering Sciences, Kyushu University, Fukuoka 816-8580, Japan*

²⁾ *Faculty of Education, Nagasaki University, 1-14 Bunkyo-machi, Nagasaki 852-8521, Japan*

³⁾ *Department of Computer and Information Sciences, Sojo University, Kumamoto 860-0082, Japan*

⁴⁾ *Research Center for Plasma Turbulence, Kyushu University, Fukuoka 816-8580, Japan*

⁵⁾ *Research Institute for Applied Mechanics, Kyushu University, Fukuoka 816-8580, Japan*

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Ionization waves exhibiting chaotic oscillations were periodized by modulating the discharge voltage. The oscillations were made periodic by applying modulation to the discharge voltage using a square wave. Furthermore, the dynamic behavior when the duty cycle of the square wave was varied was investigated. The behavior of the transition threshold separating the chaotic and periodic states was investigated. A difference was confirmed between the modulation value causing the transition from the chaotic state to the periodic state when the amplitude of the modulation voltage (square wave) was increased, and the modulation value causing the transition from the periodic to the chaotic state when the amplitude was decreased. The degree of periodization of the orbit was quantitatively evaluated using the largest Lyapunov exponent and the CH diagram, which confirmed the transition from the chaotic to the periodic state. Furthermore, chaotic periodization was possible with a duty cycle close to 50%.

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1. Introduction

Nonlinear phenomena, such as plasma chaos, have been actively researched for more than 50 years [1–9]. Chaos is a typical nonlinear phenomenon, and research on chaos control [10–14] and chaotic phenomena in plasma has been conducted. Ionization waves [15–20] produced by the ionizing instability in a positive column under glow discharge have been treated as a medium for experimental research in nonlinear physics. Nonlinear oscillations, such as chaos and limit cycles, have been observed by the optical measurement of ionization waves. Previous studies have addressed the regularization of chaotic time series by applying periodic external modulation using the frequency and amplitude as control parameters. Thus, chaotic behavior can be controlled by modulation, with the fundamental frequency of the system exhibiting chaos and an increasing applied voltage.

In this study, we demonstrate the existence of an additional control parameter for chaos control by applying a square wave instead of a periodic sinusoidal wave as an external modulation to an oscillation in a chaotic state in ionization waves.

Thus, the duty cycle of the square wave serves as an additional control parameter. Our experiments revealed an optimized duty cycle of approximately 50% with a slight skewness toward higher values. In addition, a sinusoidal wave was applied as an external modulation, and the square and sinusoidal wave cases were compared.

The remainder of this paper is organized as follows. First, the experimental setup is described. Second, the experimental results are discussed, focusing on the changes in the system before and during the application of external modulation, the control behavior when the duty cycle is changed, and the CH diagram. Finally, the conclusions of this study are presented.

2. Experimental Setup

Figure 1 shows a schematic of the experimental setup. The experiment was performed using a glass tube with a diameter and length of 2.0 and 75 cm, respectively. The glass tube was evacuated to high vacuum and then filled with neon gas at a pressure of approximately 478 Pa. When a DC electric field was applied between electrodes placed 60 cm apart in the glass tube using a DC high-voltage power supply (HV1.5-0.3, TAKASAGO), neon plasma was generated by

*Corresponding author's e-mail: h.miyoshi@riam.kyushu-u.ac.jp

**Corresponding author's e-mail: fukuyama-takao@nagasaki-u.ac.jp

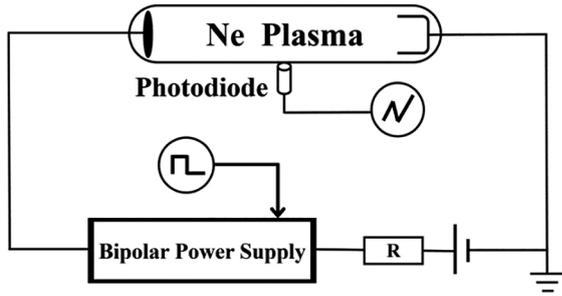


Fig. 1. Schematic of the experimental setup.

glow discharge, and ionization waves were excited owing to ionization instability. Various nonlinear phenomena, such as chaos and limit cycles, were observed in the positive column of the ionization wave as the current value changed.

This instability can be attributed to ionization collisions of neutral particles in the neon plasma, namely, instabilities associated with the ionization and excitation processes and the loss mechanisms of excited atoms. As a consequence of this instability, self-excited oscillations are generated, which are characterized by the propagation of the ionization wave-front along the positive column. Measurement of the fluctuations in light intensity reveals moving striations at several kilohertz. Ionization waves can be modeled by focusing on the phenomenon of spontaneous rhythms occurring within positive columns and employing the van der Pol equation, which exhibits self-sustaining oscillations [21].

Figure 2 shows the ionization wave in the quasiperiodic state ($I = 22.4$ mA). The power spectrum (a), wavelength spectrum (b), and dispersion relation (c) based on them are shown. The phase velocity of the fundamental wave was approximately 140 m/s. Considering the dispersion relation, a characteristic feature observed in ionization wave experiments is that the phase velocity and group velocity often become oppositely directed, forming a backward wave relationship. Consequently, as shown in Fig. 2, the dispersion relation is constructed from graphs of the power spectrum and spatial spectrum, which are created using data from the ionization wave experiments.

Time-series signals for the analysis were obtained from the fluctuation components of the light intensity from the positive column of the ionization waves using a photodiode (S6775, HAMAMATSU) and the LabVIEW system (2020, NI). This photodiode was chosen because its sensitive range from visible to near-infrared light (350–1,100 nm) is well suited for capturing the principal light output from neon. No filters were used, which allowed the recording of the total light intensity across the entire sensitive range of the sensor. This method provides an effective record of light emission. The signal output from a function generator (33220A, AGILENT) was applied to the ionization wave system using a bipolar power supply (BWS 60-5, TAKASAGO). The circuit included a 4.7 kΩ resistor to maintain a stable glow discharge.

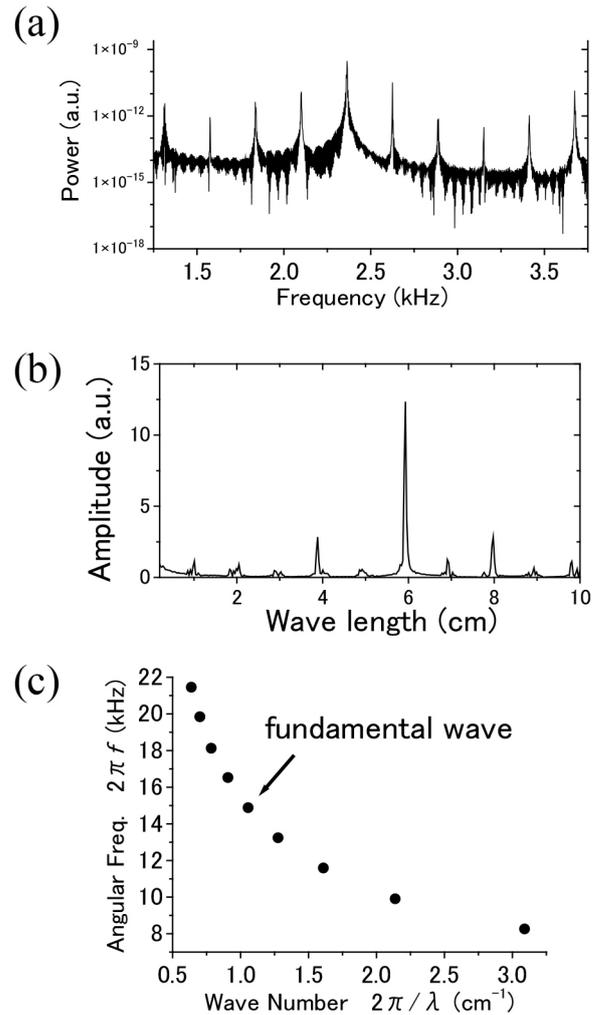


Fig. 2. Ionization wave ($I = 22.4$ mA). (a) Power spectrum; (b) wavelength spectrum; and (c) dispersion relation based on them.

3. Results and Discussions

In this study, periodic external modulation was applied to a system that exhibited a chaotic state to periodize its oscillations. During the experiment, the discharge current was maintained at 19.6 mA. Figure 3 shows the time series, reconstructed trajectory in phase space, and power spectrum before (Fig. 3(a)) and during (Fig. 3(b)) the application of external modulation. A square wave with a duty cycle of 55%, frequency of 2.429 kHz, and a voltage of 2.8 V that was generated by a function generator and applied to the ionization wave via a bipolar power supply, served as an external modulation. Trajectories in the phase space were reconstructed using the delayed embedding method based on Packard’s approach [22]. The dynamic behavior of dissipative systems is characterized by trajectories constructed in the phase space; however, it is necessary to reconstruct these trajectories in the phase space using experimental data. The method proposed by Packard was employed to reconstruct the trajectories in the phase space by incorporating a time delay. For time-series data denoted as $X(t)$ and a time delay value denoted as τ , a

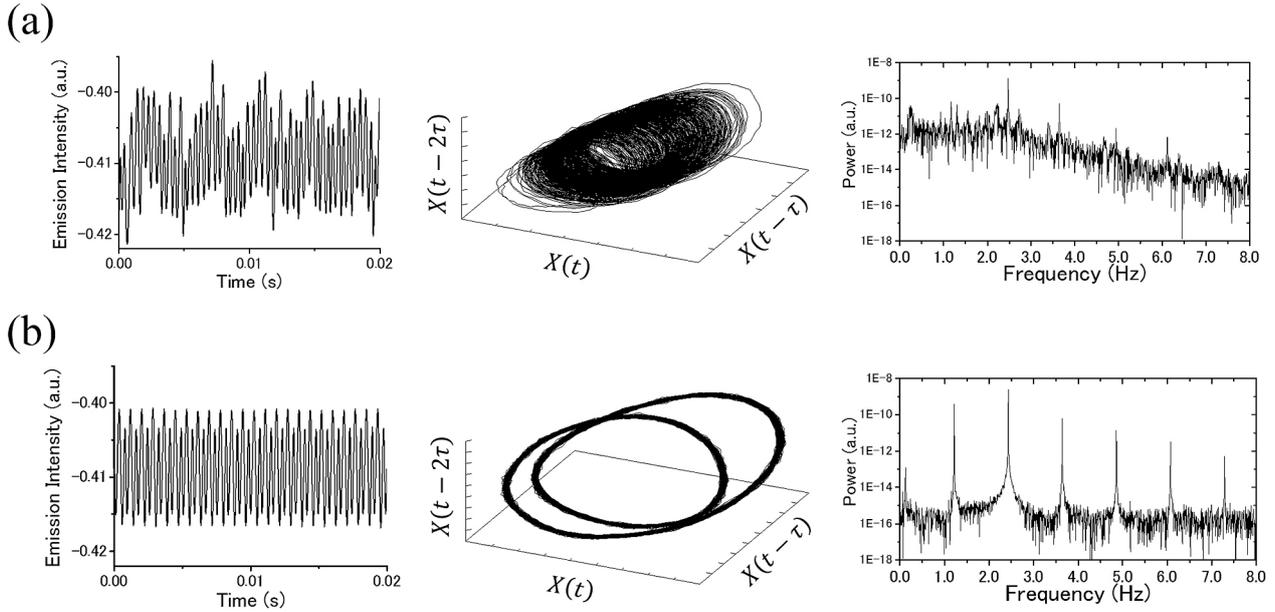


Fig. 3. Time series, reconstructed trajectory in phase space, and power spectrum (a) before and (b) during the application of the external modulation.

three-dimensional orbit can be reconstructed by plotting the values of $X(t)$, $X(t - \tau)$, and $X(t - 2\tau)$ on the axes. Here, the value of τ is typically chosen as a fraction of the fundamental period.

As shown in Fig. 3(a), the chaotic state with a broad spectrum transitions to a periodic state with a sharp spectrum upon the application of square external modulation (Fig. 3(b)). The reconstructed trajectories in the phase space form simple loops in periodic states (limit cycles) and exhibit patterns that fill the phase space during chaotic states. Accordingly, it can be observed that applying a square wave causes the chaotic trajectory to transition into a periodic one.

Figure 4 shows the largest Lyapunov exponents [23] before and during the application of the external modulation. For an initial difference denoted as $\epsilon(0)$, this difference increases exponentially. The difference $\epsilon(t)$ after time t has elapsed can be expressed as $\epsilon(t) = \epsilon(0) \times \exp(\lambda t)$. This λ represents the increase rate of the difference and is called the Lyapunov exponent. Its dimensions are the inverse of those of time. When the Lyapunov exponent is calculated from the experimental data, it is derived from trajectories in the three-dimensional phase space. Consequently, three Lyapunov exponents are calculated, and the largest among them is called the largest Lyapunov exponent [24]. The larger this largest Lyapunov exponent is, the greater the rate of increase in the initial difference becomes, indicating a more chaotic state. Three dimensions were chosen as the embedding dimensions. Furthermore, for calculation convenience, the data were thinned out equally. The closer the value of the exponent is to 0, the more periodic the state becomes, whereas the larger the value is, the more disturbed the state becomes. Consequently, the largest Lyapunov exponent was approximately 0.348 before application (Fig. 3(a)) and approximately 0.0214 during application (Fig. 3(b)). Thus, the chaotic system was

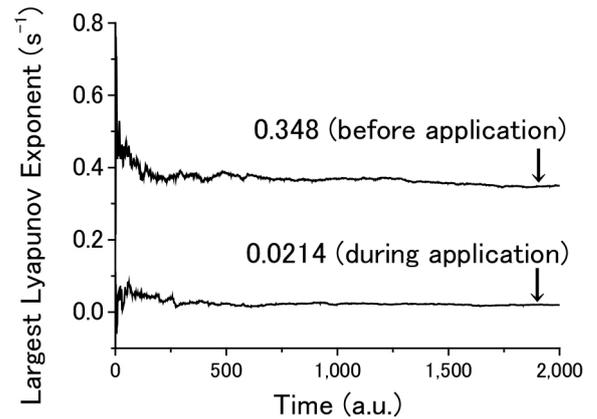


Fig. 4. Largest Lyapunov exponent before and during the application of the external modulation.

quantitatively determined to be periodized by applying square external modulation. The above results show that applying a square wave causes chaos to transition to a periodic orbit.

Controlling the plasma fluctuations in real time will be examined in a future study using cost-effective analytical methods to quantify the fluctuations. Therefore, an attractive approach based on complexity and entropy [25], that is, CH analysis, has been proposed. CH analysis was applied to the system before and during the application of external modulation, treating a part of the time-series data as a pattern and statistically evaluating the pattern included in the data using Shannon entropy. Specifically, two statistics, permutation entropy (H) and statistical complexity (C), were computed from the time-series data to characterize the dynamic behavior on the CH diagram, where a set of (H, C) is permitted in the gray region shown in Fig. 5. Specifically, low H and C indicate highly periodic behavior (typically $H \sim 0.5$ and $C \sim 0.45$), whereas high H and low C correspond to random noise

($H > 0.9$). Please note that H and C are statistics derived from probability distributions; therefore, they are dimensionless. Intermediate H , which has a relatively high C value, represents chaotic or structurally complex dynamics.

Figure 5 shows the CH diagrams before and during the application of external modulation based on CH analysis. Figure 5 shows that the value of H is large in the system in a chaotic state before the application of external modulation. However, the value of H was small and the value of C was large in the system in a periodic state during the application of external modulation. Thus, the changes before and during the application of external modulation were qualitatively distinguished.

A square wave (duty cycle: 55%, amplitude: 2.8 V, frequency: 2.429 kHz) was applied as the external modulation,

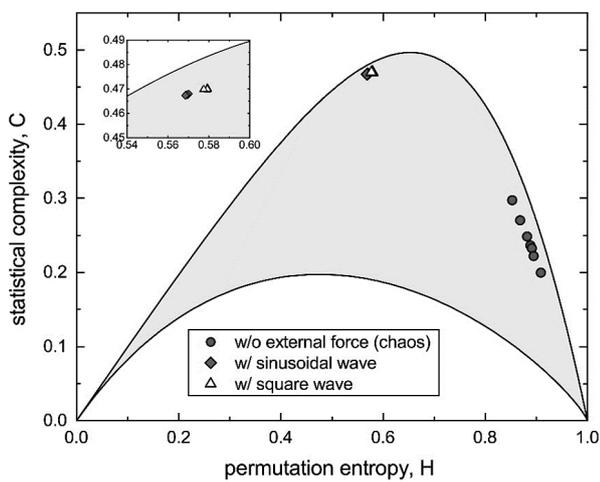


Fig. 5. CH diagram before and during the application of the external modulation.

and a sinusoidal wave with an amplitude of 3.5 V and a frequency of 2.429 kHz was applied. Consequently, the CH diagram clearly changed compared to before the force was applied, regardless of whether the external force is a sinusoidal or square wave.

A parametric scan of the duty cycle was performed. Figure 6 shows the threshold modulation voltage intensity, separating the chaotic and periodic states, as a function of the duty cycle of the square wave. The lower and upper regions of the figure represent the chaotic and periodic states, respectively. The rectangles and circles in Fig. 6 denote the cases of increasing and decreasing voltages, respectively, while the error bars indicate the standard deviation. “Increasing voltage” refers to the experimental procedure through which the discharge modulation voltage is gradually increased, causing the system to transition from a chaotic state to a periodic state upon reaching a certain threshold, which is plotted using rectangle markers (■); conversely, “decreasing voltage” denotes the procedure where the modulation voltage is gradually decreased, causing the system to transition from an initial periodic state to a chaotic state upon reaching its own threshold, which is plotted using circular markers (●). Comparing cases where the modulation voltage is increased versus decreased, each marker (■ and ●) exhibits different values, indicating that the threshold values for transitioning between chaotic and periodic states differ. In addition, larger modulation voltages are required to periodize the orbit when the duty cycle departs from the center (50%). It was found that within the range of 40% to 75% duty cycle for the square wave, chaos can be periodized using a smaller modulation voltage than when using a sinusoidal wave.

Figure 7 shows a plot of the standard deviation (error bar) of the graph in Fig. 6 against the duty cycle of the

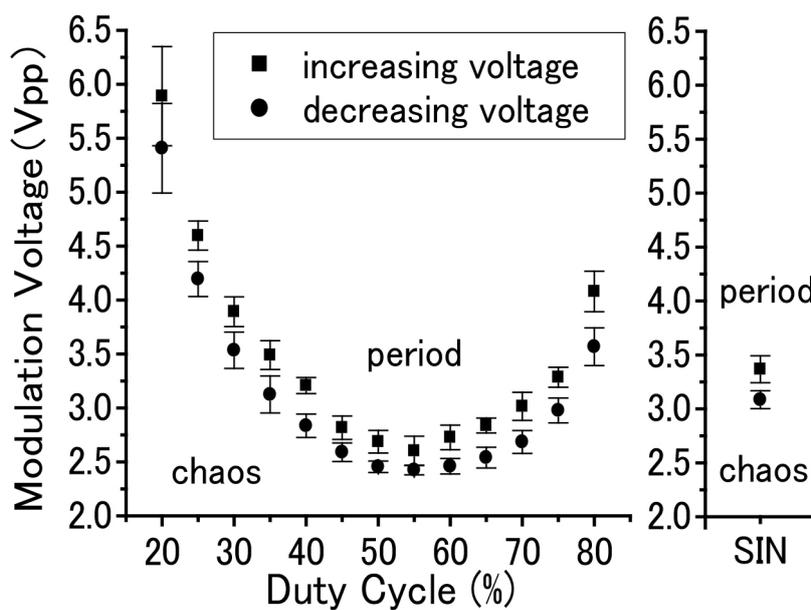


Fig. 6. Intensity of the modulation voltage that serves as the threshold between the chaotic and periodic states when the duty cycle in a square wave is varied. The lower and upper regions show the chaotic and periodic states, respectively. The rectangular and circular symbols indicate increasing and decreasing voltages, respectively.

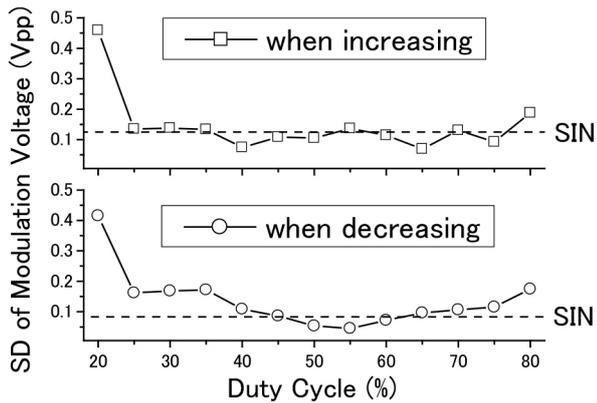


Fig. 7. Standard deviation (error bar) of the graph in Fig. 6 versus duty cycle of the square wave. The upper and lower traces represent the standard deviation of the threshold values measured multiple times as the modulation voltage between the chaotic and periodic states is increased and decreased, respectively.

square wave. The standard deviation is larger when the duty cycle departs from the center (50%). The upper and lower traces in Fig. 7 represent the standard deviations of the threshold values measured multiple times as the modulation voltage between the chaotic and periodic states increased and decreased, respectively. In each graph, the dotted line indicates the standard deviation when a sinusoidal wave is applied. This result shows that when the duty cycle is low (approximately 20%), the threshold modulation voltage required for the system to transition between chaotic and periodic states upon square-wave application is unstable and highly variable. In contrast, when the duty cycle is middle-range (approximately 50%), the transition threshold between the chaotic and periodic states is stable and exhibits consistent values.

4. Conclusion

The periodization of chaotic orbits observed in ionization waves by the application of square waves was studied experimentally and via chaos analysis.

The instability of ionization waves represents the instability in the ionization/excitation and decay processes of excited atoms. They exhibit a strong nonlinearity, potentially resulting in diverse nonlinear phenomena. The physical quantity measured is the fluctuation in the optical intensity of the ionization wave, which is observed as bright and dark stripes at frequencies of several kilohertz.

Using the largest Lyapunov exponent and CH diagrams for evaluation, it was quantitatively demonstrated that apply-

ing voltage modulation causes chaos to transition to a periodic state. The threshold for the chaos-to-periodic state transition (achieved by increasing the square-wave modulation of the discharge voltage) is different from that for the reverse periodic-to-chaos transition (achieved by decreasing that modulation), illustrating hysteresis. When the duty cycle is middle-range (approximately 50%), the transition thresholds between chaotic and periodic states are stable and exhibit consistent values. Furthermore, the chaotic behavior can be periodized using a modulation voltage smaller than that required for a sinusoidal wave. These results have significant implications for precise control of fluctuating plasmas when square waves are used as alternatives to sinusoidal waves.

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