Lagrangian Description of Reduced Magnetohydrodynamics

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This Letter presents a Lagrangian formulation of reduced magnetohydrodynamics (RMHD) for Alfvénic fluctuations in a uniform background magnetic field. The RMHD equations are derived in Lagrangian coordinates through the least action principle. We also demonstrate that cross helicity conservation is naturally tied to fluid element relabeling symmetry.

Keywords: magnetohydrodynamics, lagrangian description, Alfvén waves

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There are two distinct ways to describe hydro and magnetohydrodynamics (MHD): Eulerian and Lagrangian descriptions. In the former, one observes how fluid properties (such as density, velocity, and so on) evolve at fixed positions $\mathbf{x}$, while in the latter, one follows individual fluid elements identified by the labels $\mathbf{a}$ and tracks the evolution of their properties. For inviscid flows, these two descriptions are interchangeable via the Euler-Lagrange map $\mathbf{x} = \mathbf{q}(t, \mathbf{a})$. The label $\mathbf{a}$ is typically interpreted as the initial position of a fluid element, i.e., $\mathbf{q}(0, \mathbf{a}) = \mathbf{a}$.

Although many theoretical studies and major numerical MHD codes (*e.g.*, [1–3]) rely on the Eulerian description, the Lagrangian description has several notable advantages. First, it is theoretically elegant, offering a unified mathematical framework. In Lagrangian description, the Euler-Lagrange map $\mathbf{q}$ is the only dynamical variable; all other variables become *attributes* [4] of the fluid element. These attributes are differential forms that are Lie-dragged by the flow $\mathbf{u}(t, \mathbf{x}) = \dot{\mathbf{q}}(t, \mathbf{a})|_{\mathbf{a}=\mathbf{q}^{-1}(t,\mathbf{x})}$, where the dot denotes the time derivative at fixed $\mathbf{a}$. More specifically, thermodynamic specific entropy, magnetic flux density, and particle number density correspond to zero-, two-, and three-forms, respectively. Moreover, the dynamical equations for $\mathbf{q}$ can be derived from a least action principle, and the conservation laws, including mass, momentum, and helicity, are naturally tied to the symmetries of the action [5–8]. Second, in practical terms, Lagrangian description helps avoid the artificial violation of conservation laws that may arise due to numerical dissipation because conservation is inherently built into the formulation. As an example, Zhou *et al.* used this approach to study singular current sheets, which are otherwise prone to artificial reconnection in the conventional approach [9–11].

The application of Lagrangian description has expanded from ideal MHD to encompass more advanced plasma models, including Hall MHD and extended MHD [12, 13], gyroviscous MHD [4], and relativistic MHD [14]. In this study, we aim to formulate the Lagrangian description of reduced MHD (RMHD) [15], assuming a strong mean magnetic field. RMHD is obtained by asymptotically expanding MHD in terms of the small parameter $\epsilon \sim k_\parallel/k_\perp \sim \delta B/B_0 \sim u/v_A$, where $k_\parallel$ and $k_\perp$ are wavenumber components parallel and perpendicular to the mean magnetic field $\mathbf{B}_0$, v_A is the Alfvén speed associated with the background fields, and u and δB represent velocity and magnetic fluctuations, respectively. RMHD is widely applied in magnetic confinement fusion plasmas, as well as in space and astrophysical plasmas [16–21]. Although RMHD can describe both Alfvénic and slow-magnetosonic fluctuations in complex mean field configurations, here we focus exclusively on Alfvénic fluctuations in a spatially uniform, straight mean magnetic field.

We first set the Eulerian coordinate $\mathbf{x} = (x^1, x^2, z)^\mathsf{T}$, where the z-direction aligns with the mean magnetic field. The set of RMHD equations in Eulerian coordinates is

$$\frac{\partial \mathbf{u}_\perp}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp \mathbf{u}_\perp = -\nabla_\perp\left(\delta p^{(2)} + \frac{|\delta \mathbf{B}_\perp|^2}{2}\right) + \delta\mathbf{B}_\perp \cdot \nabla_\perp \delta\mathbf{B}_\perp + B_0 \frac{\partial \delta\mathbf{B}_\perp}{\partial z}, \tag{1a}$$

$$\frac{\partial \delta\mathbf{B}_\perp}{\partial t} + \mathbf{u}_\perp \cdot \nabla_\perp \delta\mathbf{B}_\perp = \delta\mathbf{B}_\perp \cdot \nabla_\perp \mathbf{u}_\perp + B_0 \frac{\partial \mathbf{u}_\perp}{\partial z}, \tag{1b}$$

$$\nabla_\perp \cdot \mathbf{u}_\perp = 0, \tag{1c}$$

$$\nabla_\perp \cdot \delta\mathbf{B}_\perp = 0, \tag{1d}$$

where $\mathbf{u}_\perp$ and $\delta\mathbf{B}_\perp$ are the perpendicular velocity and magnetic fluctuations[#1], and $\delta p^{(2)}$ is the second order pressure (the first order is set by the pressure balance). Here, the magnetic field is rescaled by $B_0/\sqrt{4\pi\rho_0} \to B_0$ and $\delta\mathbf{B}/\sqrt{4\pi\rho_0} \to \delta\mathbf{B}$.

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We also assume there are no parallel components of velocity or magnetic fluctuations, as we focus on Alfvén waves.

Since fluid elements move only in the perpendicular plane, we set the Lagrangian coordinate $\mathbf{a} = \left(a^1, a^2, a^3 = z\right)^{\mathsf{T}}$. The Euler-Lagrange map is

$$x^i = q^i\left(\tau, a^1, a^2, a^3\right) \quad (i = 1, 2), \tag{2a}$$

$$z = q^3\left(a^3\right) = a^3, \tag{2b}$$

where τ is the Lagrangian time variable that gives $\partial/\partial\tau = \partial/\partial t + \mathbf{u}_\perp \cdot \nabla_\perp$. The perpendicular components of the map, $\mathbf{q}_\perp = \left(q^1, q^2, 0\right)^{\mathsf{T}}$, are $O(\epsilon)$ because the perpendicular velocity fluctuation $\mathbf{u}_\perp = \dot{\mathbf{q}}_\perp$ must be $O(\epsilon)$. Hereafter, Latin indices refer to 1 and 2, and Greek indices to 1, 2, and 3. Because of the perpendicular incompressibility condition Eq. (1c), the Jacobian $\mathcal{J} = \det\left(\partial q^\alpha/\partial a^\beta\right)$ remains unity. We also introduce A^α_β, the cofactor of $\partial q^\beta/\partial a^\alpha$, which helps transform between Lagrangian and Eulerian variables:

$$\delta^\alpha_\beta = A^\gamma_\beta \frac{\partial q^\alpha}{\partial a^\gamma}, \tag{3}$$

where the left-hand side is the Kronecker delta. See Refs. [7, 14, 22] for further useful formulae.

Next, we define magnetic fields in Lagrangian coordinate $\mathcal{B} = (\mathcal{B}^1, \mathcal{B}^2, \mathcal{B}^3)^{\mathsf{T}}$. Generally, Lagrangian and Eulerian magnetic fields are interchangeable via $B^\alpha = \mathcal{B}^\beta\left(\partial q^\alpha/\partial a^\beta\right)$. For Lagrangian coordinate defined above, one obtains

$$\delta {B_\perp}^i = \mathcal{B}^\alpha \frac{\partial q^i}{\partial a^\alpha} \quad (i = 1, 2), \tag{4a}$$

$$B_0 = \mathcal{B}^\alpha \frac{\partial q^3}{\partial a^\alpha} = \mathcal{B}^3. \tag{4b}$$

We, then, make two assumptions for $\mathcal{B}^\alpha$; first we assume

$$\frac{\partial \mathcal{B}^\alpha}{\partial a^\alpha} = 0. \tag{5}$$

This is equivalent to Eq. (1d) in Eulerian coordinates because

$$\begin{aligned} 0 \;&=\; \frac{\partial \mathcal{B}^\alpha}{\partial a^\alpha} = \frac{\partial}{\partial x^\alpha}\left(\mathcal{B}^\beta \frac{\partial q^\alpha}{\partial a^\beta}\right) \\ &= \frac{\partial}{\partial x^i}\left(\mathcal{B}^j \frac{\partial q^i}{\partial a^j} + B_0 \frac{\partial q^i}{\partial a^3}\right) \\ &\quad + \frac{\partial}{\partial z}\left(\mathcal{B}^j \frac{\partial q^3}{\partial a^j} + B_0 \frac{\partial q^3}{\partial a^3}\right) \\ &= \nabla_\perp \cdot \delta\mathbf{B}_\perp. \end{aligned} \tag{6}$$

The second assumption is $\dot{\mathcal{B}}^\alpha = 0$, which yields the induction equation, Eq. (1b), in Eulerian coordinates:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\delta B^i_\perp \;&=\; \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\mathcal{B}^j \frac{\partial q^i}{\partial a^j} + B_0 \frac{\partial q^i}{\partial a^3}\right) \\ &= \mathcal{B}^j \frac{\partial q^\alpha}{\partial a^j}\frac{\partial \dot{q}^i}{\partial x^\alpha} + B_0 \frac{\partial q^\alpha}{\partial a^3}\frac{\partial \dot{q}^i}{\partial x^\alpha} \\ &= \left(\mathcal{B}^j \frac{\partial q^k}{\partial a^j} + B_0 \frac{\partial q^k}{\partial a^3}\right)\frac{\partial \dot{q}^i}{\partial x^k} + B_0 \frac{\partial \dot{q}^i}{\partial x^3} \\ &= \delta\mathbf{B}_\perp \cdot \nabla_\perp u^i_\perp + B_0 \frac{\partial u^i_\perp}{\partial z}. \end{aligned} \tag{7}$$

We now construct the action for RMHD:

$$S = \int \mathrm{d}^3\mathbf{a}\left[\frac{\dot{q}^i \dot{q}_i}{2} \;-\; \frac{B_0^2}{2\mathcal{J}}\frac{\partial q^\alpha}{\partial a^3}\frac{\partial q_\alpha}{\partial a^3} \;+\; \lambda(\mathcal{J} - 1)\right]. \tag{8}$$

The last term enforces incompressibility $\mathcal{J} = 1$, and λ is a Lagrange multiplier[#2] [23]. This action reduces to the $O(\epsilon^2)$ terms of the incompressible full-MHD action [22]. Varying S with respect to q^i (noting that $\delta q^3 = 0$) yields the equation of motion:

$$\ddot{q}_i = \frac{\partial}{\partial a^\alpha}\left[\mathcal{B}^\alpha \mathcal{B}^\beta \frac{\partial q_i}{\partial a^\beta} - A^\alpha_i\left(\frac{\mathcal{B}^\gamma \mathcal{B}^\delta}{2}\frac{\partial q^\beta}{\partial a^\gamma}\frac{\partial q_\beta}{\partial a^\delta} + \lambda\right)\right]. \tag{9}$$

This is the RMHD equation of motion in Lagrangian coordinates. Transforming to Eulerian coordinates, the left hand side becomes $\partial_t \mathbf{u}_\perp + \mathbf{u}_\perp \cdot \nabla_\perp \mathbf{u}_\perp$, while the right side becomes $B_0\,\partial_z \delta\mathbf{B}_\perp + \delta\mathbf{B}_\perp \cdot \nabla_\perp \delta\mathbf{B}_\perp - \nabla_\perp\left(\lambda + |\delta\mathbf{B}_\perp|^2/2\right)$. This matches Eq. (1a), except for λ, which must be equal to the pressure $\delta p^{(2)}$ determined by the condition $\nabla_\perp \cdot \mathbf{u}_\perp = 0$ [22, 23].

Next, we derive the conservation of helicity and demonstrate that it stems from the fluid element relabeling symmetry. The derivation is essentially the same as that in Refs. [5–7]. When the equation of motion is satisfied, a general transformation $\tau \to \tau'$, $\mathbf{a} \to \mathbf{a}'$, $\mathbf{q}(\tau, \mathbf{a}) \to \mathbf{q}'(\tau', \mathbf{a}')$ induces the change in action

$$\begin{aligned} \delta S \;=\; \int \mathrm{d}\tau \int \mathrm{d}^3\mathbf{a} \Bigg\{ &\frac{\partial}{\partial \tau}\left(\mathcal{L}\delta\tau + \delta q^\alpha \frac{\partial \mathcal{L}}{\partial \dot{q}^\alpha}\right) \\ &+ \frac{\partial}{\partial a^\alpha}\left[\mathcal{L}\delta a^\alpha + \delta q^\beta \frac{\partial \mathcal{L}}{\partial\left(\partial_\alpha q^\beta\right)}\right]\Bigg\}, \end{aligned} \tag{10}$$

where $\delta\tau = \tau' - \tau$, $\delta a^\alpha = a'^\alpha - a^\alpha$, and $\delta q^\alpha = q'^\alpha(\tau', \mathbf{a}') - {}-q^\alpha(\tau, \mathbf{a}) - \delta a^\beta\left(\partial q^\alpha/\partial a^\beta\right)$. We then consider a specific transformation

$$\tau' - \tau = 0, \tag{11a}$$

$$a^{i\prime} - a^i = \mathcal{B}^i, \tag{11b}$$

$$a'^3 - a^3 = 0, \tag{11c}$$

$$q'^\alpha(\tau', \mathbf{a}') - q^\alpha(\tau, \mathbf{a}) = 0, \tag{11d}$$

which is a relabeling transformation, since only the label a^i is shifted. Applying this to the RMHD action Eq. (8) and invoking the equation of motion Eq. (9), the integrand becomes

$$\begin{aligned} &\frac{\partial}{\partial \tau}\left(\mathcal{B}^\alpha \frac{\partial q^i}{\partial a^\alpha}\dot{q}_i\right) \\ &- \frac{\partial}{\partial a^\alpha}\Bigg[\left(\frac{\dot{q}^i \dot{q}_i}{2} + \frac{\mathcal{B}^\gamma \mathcal{B}^\delta}{2}\frac{\partial q^\beta}{\partial a^\gamma}\frac{\partial q_\beta}{\partial a^\delta}\right)\mathcal{B}^\alpha \\ &- \mathcal{B}^\beta \frac{\partial q^i}{\partial a^\beta}\left(\frac{\mathcal{B}^\delta \mathcal{B}^\varepsilon}{2}\frac{\partial q^\gamma}{\partial a^\delta}\frac{\partial q_\gamma}{\partial a^\varepsilon} + \lambda\right)A^\alpha_i\Bigg] = 0, \end{aligned} \tag{12}$$

representing the local conservation of cross helicity in Lagrangian coordinates. Because the integrand of Eq. (10) is zero, the action remains invariant under the relabeling transformation Eq. (11). Finally, transforming Eq. (12) to Eulerian coordinates yields

$$
\begin{aligned}
&\frac{\partial}{\partial t}(\mathbf{u}_\perp \cdot \delta\mathbf{B}_\perp) \\
&+\nabla_\perp \cdot \left[(\mathbf{u}_\perp \cdot \delta\mathbf{B}_\perp)\,\mathbf{u}_\perp - \left(\frac{|\mathbf{u}_\perp|^2}{2} - \lambda \right) \delta\mathbf{B}_\perp \right] \\
&-\frac{\partial}{\partial z}\left(\frac{|\mathbf{u}_\perp|^2}{2} + \frac{|\delta\mathbf{B}_\perp|^2}{2} \right) = 0,
\end{aligned} \tag{13}
$$

which also follows directly from the RMHD equations in Eulerian form Eq. (1).

In summary, we have formulated the Lagrangian description of RMHD Eq. (9) and shown that the conservation of cross helicity Eq. (12) is tied to the fluid element relabeling symmetry. While we have shown only the conservation of cross helicity, the other invariants, which were systematically derived in Ref. [24], may also be derived via the relabeling symmetry (note that all the invariants of ideal MHD have been shown to be related to the relabeling symmetry [7]). For analytical tractability, we focused on Alfvénic fluctuations by neglecting slow-magnetosonic waves ($u_\parallel = \delta B_\parallel = 0$), as including these modes would introduce additional complexity because q^3 is not fixed, and $\partial/\partial\tau$ would include $u_\parallel(\partial/\partial z)$, which is a higher-order term. However, even with this simplification, the Lagrangian description presented in this work provides valuable insights into the dynamics of current sheets in the presence of a mean magnetic field.

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Note

#1 Usually, one uses a stream function $\mathbf{u}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \Phi$ and a magnetic flux function $\delta\mathbf{B}_\perp = \hat{\mathbf{z}} \times \nabla_\perp \Psi$ in RMHD.

#2 See Ref. [22] for an in-depth discussion of incompressibility in variational formalisms.

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