Predicting Spatio-Temporal Dynamics in Multi-Dimensional Turbulent Flows via Hankel Sparsity-Promoting Dynamic Mode Decomposition

Naoya MITSUHASHI, Makoto SASAKI*, Takumi KODAHARA

College of Industrial Technology, Nihon University, Narashino 275-8575, Japan

(Received 15 September 2025 / Accepted 31 October 2025)

In this study, we apply Hankel Sparsity-Promoting Dynamic Mode Decomposition (Hankel-SP-DMD) to dynamic turbulent flows in two-dimensional space to predict their long-term spatial structures. The target turbulence data are obtained from numerical simulations based on the extended Hasegawa-Wakatani model. The proposed method successfully extracts dominant modes and predicts their temporal evolution. We further evaluate the impact of hyperparameter settings required for training on prediction performance, and provide guidance for appropriate parameter selection based on the correlation with turbulence and limit-cycle periods.

Keywords: DMD, sparse modeling, plasma turbulence, long-term prediction

DOI: 10.1585/pfr.20.1201059

In magnetically confined plasmas, abrupt transport of particles and heat can damage device walls, making its prediction a critical issue [1]. Plasma turbulence driven by spatial gradients generates global structures through nonlinear interactions, but a general predictive methodology has not been well established [2]. Neural network approaches such as recurrent neural networks (RNNs) and long short-term memory (LSTM) models are being explored for time-series prediction in plasma physics [3, 4], though they often lack physical interpretability. In contrast, Dynamic Mode Decomposition (DMD) [5], a data-driven method to extract spatio-temporal structures, is increasingly applied to plasma turbulence [6–8]. Notably, Hankel sparsity-promoting DMD (Hankel-SP-DMD), which incorporates time-delay embedding and sparsity, has been shown to enable one-dimensional time-series prediction of turbulent data [7]. Since plasma turbulence exhibits quasi-two-dimensional characteristics, spatio-temporal prediction in two-dimensional fields is essential. In this study, we apply Hankel-SP-DMD to high-dimensional data including two-dimensional fields, and demonstrate its capability for predicting spatio-temporal structures. The method is applied to numerically generated turbulence governed by the Kelvin-Helmholtz instability in a cylindrical plasma model, achieving long-term prediction of turbulent structures with two-dimensional limit-cycle oscillations [9, 10]. We also evaluate the impact of hyperparameters in constructing the Hankel matrix and verify the stability of the prediction.

In this study, we apply Hankel-SP-DMD to dense data $N(r,\theta,t) \in \mathbb{R}^{N_r \times N_\theta \times N_t}$, where, N_r, N_θ, N_t denote the number of grid points in the radial, poloidal, and temporal direction, respectively. Here, we demonstrate spatio-temporal prediction in two-dimensional space (r,θ). At each time t_j, the spatial field $N(r,\theta,t_j)$ is reshaped into a one-dimensional vector $x_{t_j} = vec(N(r,\theta,t_j)) \in \mathbb{R}^{N_r N_\theta}$, where $vec(\cdot)$ denotes unfolding into a column vector. By concatenating these vectors, we construct $X = [x_{t_1}, x_{t_2}, ..., x_{t_{N_t}}] \in \mathbb{R}^{(N_r N_\theta) \times N_t}$. Applying time-delay embedding to X, we construct a block Hankel matrix [7, 11]. Here, $X(t_i, t_j)$ denotes the sequence of numerical data of $X(t)$ for $t_i \le t \le t_j$.

$$X_H := \begin{bmatrix} X(t_1, t_1 + (m-1)s) \\ X(t_2, t_2 + (m-1)s) \\ \vdots \\ X(t_d, t_d + (m-1)s) \end{bmatrix} \in \mathbb{R}^{(N_r N_\theta d) \times m}. \quad (1)$$

Here, $m = [(N_t - d)/s] + 1$ denotes the number of columns, where s is the slide width, and d is the stack size, which serve as hyperparameters controlling accuracy and cost. The row (temporal) direction of the Hankel matrix is designed to sufficiently include the time scales associated with limit-cycle behavior exhibited by the system. DMD is applied to the Hankel matrix [12] to decompose spatial modes and their temporal evolution. Sparsity is introduced by L1-regularization, which extracts only dominant modes. The optimization problem is formulated as,

$$\min_{\{\alpha_i\}} \mathrm{J}(\{\alpha_i\}) + \Upsilon \sum_{i=1}^{(N_r N_\theta d)} |\alpha_i|, \quad (2)$$

where $\mathrm{J}(\{\alpha_i\})$ denotes the reconstruction error, Υ is the L1-regularization parameter, and α_i represents the mode amplitudes.

*Corresponding author's e-mail: sasaki.makoto@nihon-u.ac.jp

These parameters are optimized to reduce noise and improve long-term prediction [7, 12].

We describe the dataset used for analysis. The two-dimensional turbulence data are obtained from simulations with the Numerical Linear Device (NLD) code, which is based on a reduced fluid model of a cylindrical plasma [9, 13, 14]. In this study, we calculate a KH turbulence field with limit-cycle oscillations [9]. The electrostatic potential distribution in Fig. 1 (left) exhibits a spiral structure, demonstrating dynamic and spatially complex turbulence suitable for evaluating prediction methods. By applying time-delay embedding to the time-series data, Hankel-SP-DMD predicts two-dimensional turbulent patterns. In this prediction, we used the following time-delay embedding hyperparameters: slide width s = 13 and number of stacks d = 3. Sparsity was controlled by the L1-regularization coefficient Υ = 5. Figure 1 (right) shows reconstructed data from the learned DMD modes and the Hankel matrix. The horizontal axis denotes time steps t normalized by the ion cyclotron frequency, while the vertical axis Z denotes a one-dimensional index formed by stacking radial data for each poloidal angle θ. The time series is divided at t = 500 into a training region (left) and a prediction region (right). The model uses data for t = 0–500 for training and predicts subsequent series. In the training region, spatial structures consistent with the original data are reconstructed, while in the prediction region the periodicity of turbulence is preserved, confirming the capability of Hankel-SP-DMD for long-term prediction with physical consistency. Figure 2 shows reconstructed two-dimensional structures compared with reference data. The snapshots at t = 552, 556 and 559 show strong agreement of spiral structures, including vortex position, rotation, and shape, particularly near the plasma edge. In contrast, the core region, in which the fluctuation level is low, a spiral structure is observed as an artifact. This confirms that reconstructed fields retain essential physical characteristics and that the method effectively captures shape variations for long-term prediction.

Figure 3 shows the mean relative error (normalized RMSE) of reconstructed and predicted data. On the horizontal axis, we plot the dimensionless time τ, defined as the Hankel (time-delay) window span $(d-1)s$ normalized by the dominant turbulence period T^*: $\tau = (d-1)s/T^*$. The dominant turbulence period T^* was estimated from the density time series (1,000 snapshots) via autocorrelation (T^* = 27.86). Thus, τ = 1 corresponds to the case including one period of history in the window. In this analysis, d was fixed, while s was varied to evaluate the prediction error. The error exhibits a transient rise initially but converges to a low value for $\tau > 1$. In this problem, since accuracy is governed by the dimensionless time τ rather than s or d, please select (s, d) that satisfy $\tau \geq 1$ for high-accuracy long-term prediction. The asymptotic level ($\approx$ 0.2) of the relative error in Fig. 3 corresponds to the DMD reconstruction-error floor (the attainable limit) for the proposed method Hankel-SP-DMD. Thus, we confirm the robustness of Hankel-SP-DMD for long-term prediction of nonlinear turbulence.

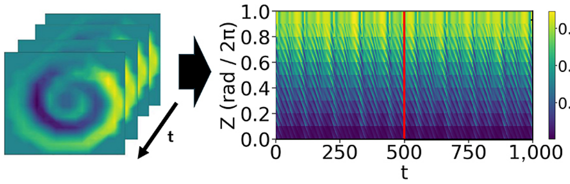

Fig. 1. Spatio-temporal evolution of two-dimensional turbulence. (Left) Two-dimensional structures in the $r-\theta$ plane. (Right) Prediction by Hankel-SP-DMD.

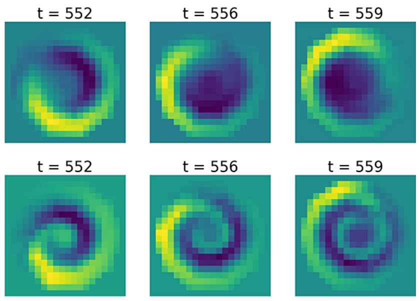

Fig. 2. Temporal snapshots of two-dimensional turbulence. (Top) Reference data. (Bottom) Prediction data obtained by Hankel-SP-DMD.

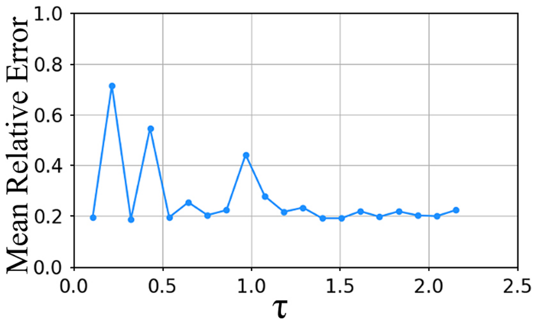

Fig. 3. Mean relative error of turbulent-structure prediction as a function of the Hankel window length. Here, τ is normalized by the turbulence period.

In this study, we apply Hankel-SP-DMD to two-dimensional turbulence data from a reduced fluid model of a cylindrical plasma and evaluate its capability for long-term prediction. The target is KH-driven turbulence with spiral structures and limit-cycle oscillations. Spatial information is vectorized and embedded into a Hankel matrix with time delays, enabling sparse extraction of dominant modes. We investigate optimal hyperparameters such as slide width, stack size, and sparsity coefficient, and demonstrate convergence of prediction errors, confirming the effectiveness of Hankel-SP-DMD for nonlinear turbulence with periodic behavior.

This work was partly supported by a grant-in-aid for scientific research of JSPS, Japan (JP21K03509, JP24K06997, JP25K00986), the collaboration programs of NIFS (NIFS23KIST044, NIFS23KIIT002, NIFS24KIPS005, NIFS23KUHL114, NIFS25KISS041), the ‘Strategic Research Projects’ Grant 2022-SRP-13 from ROIS (Research Organization of Information and Systems), and that of RIAM Kyushu University (2024S2-CD-2, 2025S2-CD-3).

The authors acknowledge discussions with Dr. Akira Kusaba of Kyushu University.

[1] J.W. Connor, Plasma Phys. Control. Fusion **40**, 191 (1998).
[2] P.H. Diamond *et al.*, Plasma Phys. Control. Fusion **47**, R35 (2005).
[3] M. AizawaCaranza *et al.*, Plasma Fusion Res. **17**, 1201048 (2022).
[4] M. Priyanka *et al.*, E3S Web Conf. **477**, 00039 (2024).
[5] P.J. Schmid, J. Fluid Mech. **656**, 5 (2010).
[6] M. Sasaki *et al.*, J. Plasma Fusion Res. **97**, 79 (2021).
[7] A. Kusaba *et al.*, Jpn. J. Appl. Phys. **61**, SA1011 (2021).
[8] M. Sasaki *et al.*, Plasma Phys. Control. Fusion **61**, 112001 (2019).
[9] M. Sasaki *et al.*, Phys. Plasmas **26**, 042305 (2019).
[10] T. Kodahara *et al.*, Plasma Phys. Control. Fusion **67**, 065012 (2025).
[11] J.H. Tu *et al.*, J. Comput. Dyn. **1**, 391 (2014).
[12] M.R. Jovanović *et al.*, Phys. Fluids **26**, 024103 (2014).
[13] N. Kasuya *et al.*, Phys. Plasmas **15**, 052302 (2008).
[14] M. Sasaki *et al.*, Phys. Plasmas **24**, 112103 (2017).