

# A Note on Stability Analysis by the Second Variations of Lagrangian and Hamiltonian for Ideal Incompressible Plasmas<sup>\*)</sup>

Keisuke ARAKI

Okayama University of Science, Okayama 700-0005, Japan

(Received 10 January 2019 / Accepted 26 February 2019)

From the viewpoint of the differential geometrical approach to the Lagrangian mechanical variational problem, a proof is presented for a general conservation law that the Lagrangian displacement type perturbation field satisfies around a stationary solution, previously derived by Hirota et al. for the Hall magnetohydrodynamic system. Additionally, this mathematical approach is applied to the Hamiltonian mechanical stability analyses of the magnetohydrodynamic and Hall magnetohydrodynamic systems.

© 2019 The Japan Society of Plasma Science and Nuclear Fusion Research

Keywords: stability analysis, Lagrangian mechanics, Hamiltonian mechanics, second variation, double Beltrami flow

DOI: 10.1585/pfr.14.3401073

## 1. Introduction

Differential geometrical approaches to the dynamics of continuous media have revealed general and fundamental mathematical structure [1, 2]. Regarding plasma physics, the dynamics of dissipationless incompressible magnetohydrodynamic (MHD) and Hall magnetohydrodynamic (HMHD) media have been commonly described as geodesics on some appropriate Lie groups [3, 4]. These dynamical systems are defined by the pair of the appropriate Riemannian metric (or inner product) and the Lie bracket of basic variables. Furthermore, Lagrangian mechanical consideration of MHD and HMHD systems revealed the relationship between the helicity conservation via Noether's theorem and the stationary force-free solutions known as double Beltrami flows [5]. In the present study, we attempt to apply these mathematical notions to the stability problems of stationary states from both the Lagrangian and Hamiltonian mechanical viewpoints.

This paper is organized as follows. Sections 2 and 3 are very abstract and based on Lie group theory and the theoretical formulation of the dynamics of wide variety of continua (see, for example, Refs. [1] or [2]). We ultimately investigate the variation problem up to the second order of perturbation from the Lagrangian mechanical viewpoint. In section 4, to illustrate the effectiveness of the general theoretical formulation, we examine the stability problems of the HMHD system and its MHD limit in the Hamiltonian mechanical perspective.

## 2. The Lagrangian Mechanics of Ideal Incompressible Fluids and Plasmas

In this section we introduce mathematical notations

author's e-mail: araki@are.ous.ac.jp

<sup>\*)</sup> This article is based on the presentation at the 27th International Toki Conference (ITC27) & the 13th Asia Pacific Plasma Theory Conference (APPTC2018).

used in the following sections. The details were discussed in Ref. [5].

Let  $\gamma(t, \epsilon)$  represent paths on the configuration space, where  $t$  and  $\epsilon$  are the time and perturbation parameters, respectively.<sup>1</sup> The generalized velocities ( $V_\epsilon(t)$ ) and the Lagrangian particle displacement fields ( $\xi(t)$ ) are defined by their partial derivatives as follows:

$$V_\epsilon(t) := \frac{\partial}{\partial t} \gamma(t, \epsilon), \quad \xi(t) := \frac{\partial}{\partial \epsilon} \gamma(t, \epsilon) \Big|_{\epsilon=0}, \quad (1)$$

and  $\gamma(t) = \gamma(t, 0)$ ,  $V := V_0$  are the respective reference path and solution.<sup>2</sup> Approximation of the paths via  $C \rightarrow D$ , and  $C \rightarrow A \rightarrow B \rightarrow D$  in terms of exponential maps<sup>3</sup> leads to the following relation:

$$e^{\tau V_\epsilon(t)} \approx e^{\epsilon \xi(t+\tau)} e^{\tau V(t)} e^{-\epsilon \xi(t)}, \quad (2)$$

where the positions of four points A:  $\gamma(t, 0)$ , B:  $\gamma(t + \tau, 0)$ , C:  $\gamma(t, \epsilon)$ , D:  $\gamma(t + \tau, \epsilon)$  are depicted in Fig. 1. The Baker-Campbell-Hausdorff formula up to the third order,

$$e^A e^B = \exp \left( A + B + \frac{1}{2} [A, B] + \frac{1}{12} ([A, [A, B]] + [B, [B, A]]) + \dots \right), \quad (3)$$

<sup>1</sup>For a perfect fluid, the path is given by the history of the Lagrangian markers:  $\gamma(t, \epsilon) = \vec{X}(\vec{a}, t; \epsilon)$ , where the initial position is given by  $\gamma(0, \epsilon) = \vec{a}$ .

<sup>2</sup>All the vector fields that appear in this study are given in the Eulerian specification. The relations between the components of the fields, ( $V_\epsilon^i$ ,  $\xi^i$ ), and those of the Lagrangian markers, ( $X^i$ ), are given as follows:

$$V_\epsilon = V_\epsilon^i \frac{\partial}{\partial x^i}, \quad \text{where } V_\epsilon^i(\vec{X}(\vec{a}, t; \epsilon), t) := \left( \frac{\partial X^i}{\partial t} \right)_{(\vec{a}, t; \epsilon)},$$

$$\xi = \xi^i \frac{\partial}{\partial x^i}, \quad \text{where } \xi^i(\vec{X}(\vec{a}, t; 0), t) = \left( \frac{\partial X^i}{\partial \epsilon} \right)_{(\vec{a}, t; 0)},$$

for a perfect fluid.

<sup>3</sup>For perfect fluid motion, an exponential map is the finite-time Lagrangian marker history generated by a constant vector field.

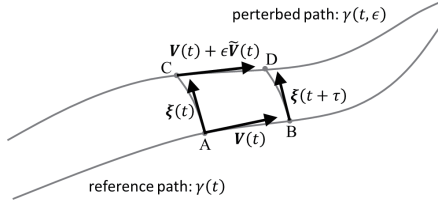


Fig. 1 The relation between the reference velocity,  $V$ , the Lagrangian displacement field,  $\xi$ , and the velocity perturbation,  $v$ , depicted as the tangent vectors of the small segments on the paths.

is used to derive the second order perturbation of the velocity field, where  $[*,*]$  is the Lie bracket of Lie algebra,<sup>4</sup> which satisfies antisymmetry and the Jacobi identity:

$$[A, B] = -[B, A], \quad (4)$$

$$[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0. \quad (5)$$

The perturbed generalized velocity due to Lagrangian displacement,  $\xi$ , up to the order  $O(\epsilon^2)$  is given by

$$\begin{aligned} V_\epsilon &:= \frac{d}{d\tau} \exp(\epsilon\xi + \epsilon\tau\dot{\xi}) \exp(\tau V) \exp(-\epsilon\xi) \Big|_{\tau=0} \quad (6) \\ &= V + \epsilon(\dot{\xi} + [\xi, V]) + \frac{\epsilon^2}{2} [\xi, \dot{\xi} + [\xi, V]] + o(\epsilon^2). \quad (7) \end{aligned}$$

The  $O(\epsilon)$  terms of  $V_\epsilon$ , which we denote by  $v$  hereafter, gives so-called Lin's constraints [2]:

$$v = \dot{\xi} + [V, \xi]. \quad (8)$$

Note that, the Lie bracket is recognizable as the Lie derivative (Ref. [6]; ch. I, sect. 3):

$$L_V \xi := \lim_{s \rightarrow 0} \lim_{t \rightarrow 0} \frac{e^{s\xi} - e^{tV} e^{s\xi} e^{-tV}}{st} = [\xi, V]. \quad (9)$$

In Arnold's differential geometrical formulation, Lagrangian is simply given by inner product of the generalized velocity,  $\mathcal{L} = \frac{1}{2} \langle V_\epsilon | V_\epsilon \rangle$ , where the bracket  $\langle * | * \rangle$  is the appropriate inner product.<sup>5</sup> Therefore, the action on the perturbed path,  $\gamma(t, \epsilon)$ , is given by

$$\begin{aligned} S_\epsilon &= \frac{1}{2} \int_0^1 dt \left\| V + \epsilon(\dot{\xi} + [\xi, V]) \right. \\ &\quad \left. + \frac{\epsilon^2}{2} [\xi, \dot{\xi} + [\xi, V]] + o(\epsilon^2) \right\|^2, \quad (10) \end{aligned}$$

where  $\|a\|^2 := \langle a | a \rangle$ . Expanding the action by  $\epsilon$ , we obtain the first variation as follows:

$$\begin{aligned} \frac{dS_\epsilon}{d\epsilon} \Big|_{\epsilon=0} &= \int_0^1 dt \langle V | \dot{\xi} + L_V \xi \rangle \\ &= \langle V | \xi \rangle \Big|_{t=0}^{t=1} - \int_0^1 dt \langle \xi | \dot{V} - L_V^\dagger V \rangle, \quad (11) \end{aligned}$$

<sup>4</sup>The Lie bracket for the ideal hydrodynamics is given by the commutator of the vector field defined by  $[a, b] = \nabla \times (a \times b) = (b \cdot \nabla)a - (a \cdot \nabla)b$ .

<sup>5</sup>The inner products are defined to give the total energies of the fluids and plasmas.

where the operator  $L^\dagger$  is defined by the following integration by parts<sup>6</sup>:

$$\langle L_b^\dagger a | c \rangle := \langle a | L_b c \rangle. \quad (12)$$

According to Hamilton's principle, the extremal condition with fixed path end points,  $(\xi(0) = \xi(1) = 0)$ , gives the Euler-Lagrange equation:

$$\dot{V} - L_V^\dagger V = 0, \quad (13)$$

which is known as the Euler-Poincaré equation [2].

### 3. The Constant of Motion Around a Stationary Solution

#### 3.1 Derivation of the constant

In this section, we consider the linear stability problem around a stationary solution to Eq. (13), which satisfies

$$\dot{V} = 0, \quad (14)$$

$$L_V^\dagger V = 0. \quad (15)$$

Substituting  $V + \epsilon v$  into the Euler equation (13) and expanding regarding the power of  $\epsilon$ , we obtain the linearized perturbation equation at the order  $O(\epsilon)$ :

$$\dot{v} = L_V^\dagger v + L_v^\dagger V. \quad (16)$$

Taking the inner product with  $\dot{\xi}$ , we obtain the following evolution equation:

$$\langle \dot{\xi} | \dot{v} \rangle = \langle \dot{\xi} | L_V^\dagger v \rangle + \langle \dot{\xi} | L_v^\dagger V \rangle. \quad (17)$$

Using the relations (4), (5), (8), (9), (12), (14), and (15), and performing the following tricky calculations, each term of Eq. (17) is rewritten as follows:

$$\langle \dot{\xi} | \dot{v} \rangle \stackrel{(8)}{=} \langle \dot{\xi} | \ddot{\xi} \rangle + \langle \dot{\xi} | L_V \dot{\xi} \rangle = \frac{1}{2} \frac{d}{dt} \langle \dot{\xi} | \dot{\xi} \rangle + \langle \dot{\xi} | L_V \dot{\xi} \rangle, \quad (18)$$

$$\begin{aligned} \langle \dot{\xi} | L_V^\dagger v \rangle &\stackrel{(12)}{=} \langle L_V \dot{\xi} | v \rangle \stackrel{(8)}{=} \langle L_V \dot{\xi} | \dot{\xi} \rangle + \langle L_V \dot{\xi} | L_V \xi \rangle \\ &\stackrel{(14)}{=} \langle L_V \dot{\xi} | \dot{\xi} \rangle + \frac{1}{2} \frac{d}{dt} \langle L_V \dot{\xi} | L_V \xi \rangle \\ &\stackrel{(4)}{=} \langle L_V \dot{\xi} | \dot{\xi} \rangle + \frac{1}{2} \frac{d}{dt} \langle L_\xi V | L_\xi V \rangle, \quad (19) \end{aligned}$$

$$\begin{aligned} \langle \dot{\xi} | L_v^\dagger V \rangle &\stackrel{(12)}{=} \langle L_v \dot{\xi} | V \rangle \stackrel{(4)}{=} -\langle L_\xi v | V \rangle \stackrel{(12)}{=} -\langle v | L_\xi^\dagger V \rangle \\ &\stackrel{(8)}{=} -\langle \dot{\xi} | L_\xi^\dagger V \rangle - \langle L_V \dot{\xi} | L_\xi^\dagger V \rangle \stackrel{(12)}{=} -\langle L_\xi \dot{\xi} | V \rangle - \langle L_\xi L_V \dot{\xi} | V \rangle \\ &\stackrel{(4)}{=} \langle L_\xi L_\xi V | V \rangle \stackrel{[A]}{=} \frac{1}{2} \langle L_\xi L_\xi V | V \rangle + \frac{1}{2} \langle L_\xi L_\xi V | V \rangle \\ &\stackrel{(14)}{=} \frac{1}{2} \frac{d}{dt} \langle L_\xi L_\xi V | V \rangle \stackrel{(12)}{=} \frac{1}{2} \frac{d}{dt} \langle L_\xi V | L_\xi^\dagger V \rangle, \quad (20) \end{aligned}$$

where the numbers on top of the equal symbols denote the equations used. Derivation of the step [A] of Eq. (20) is

<sup>6</sup>In Arnold's textbook the operator  $L^\dagger$  is denoted by  $B(*, *)$  ([1]; appendix 2, sect. B).

given by

$$\begin{aligned} \langle L_{\xi} L_{\xi} V | V \rangle &\stackrel{(5)}{=} -\langle L_{\xi} L_V \dot{\xi} | V \rangle - \langle L_V L_{\xi} \dot{\xi} | V \rangle \\ &\stackrel{(4)}{=} \langle L_{\xi} L_{\xi} V | V \rangle - \langle L_{\xi} \dot{\xi} | L_V^{\dagger} V \rangle \stackrel{(15)}{=} \langle L_{\xi} L_{\xi} V | V \rangle \\ &\stackrel{(12)}{=} \frac{1}{2} \langle L_{\xi} L_{\xi} V | V \rangle + \frac{1}{2} \langle L_{\xi} L_{\xi} V | V \rangle. \end{aligned} \quad (21)$$

Therefore, equation (17) is rewritten as

$$\frac{1}{2} \frac{d}{dt} \langle \dot{\xi} | \dot{\xi} \rangle = \frac{1}{2} \frac{d}{dt} \langle L_{\xi} V | L_{\xi} V \rangle + \frac{1}{2} \frac{d}{dt} \langle L_{\xi} V | L_{\xi}^{\dagger} V \rangle, \quad (22)$$

which leads to the following constant of motion:

$$I = \langle L_{\xi} V | L_{\xi} V \rangle + \langle L_{\xi} V | L_{\xi}^{\dagger} V \rangle - \langle \dot{\xi} | \dot{\xi} \rangle. \quad (23)$$

While the first term is positive definite, the third term is negative definite. As aforementioned in Ref. [4] in terms of sectional curvature analysis, the second term can be positive or negative.

### 3.2 Implications of the constant of motion

The constant of motion (23) is not obtained from the energy growth of the velocity<sup>7</sup>  $\langle v | \dot{v} \rangle$ , but rather the inner product  $\langle \dot{\xi} | \dot{v} \rangle$ . According to Noether's theorem, some continuous symmetry of the considered system induces conservation law [7].

The second variation of action (10) leads to the following expressions:

$$\left. \frac{d^2 S_{\epsilon}}{d\epsilon^2} \right|_{\epsilon=0} = \frac{1}{2} \int_0^1 dt \langle v | v \rangle + \langle V | [\xi, v] \rangle \quad (24)$$

$$= \frac{1}{2} \langle v | \xi \rangle \Big|_{t=0}^{t=1} - \frac{1}{2} \int_0^1 dt \langle \xi | \dot{v} - L_V^{\dagger} v - L_V^{\dagger} V \rangle. \quad (25)$$

The second equation implies that the second variation naturally induces the linearized equation (16). Remember that the constant (23) was obtained by multiplying  $\dot{\xi}$  by the linearized equation (16), producing an expression quite similar to the second term of Eq. (25).

Here, we posit that the constant is related to the infinitesimal time translation of the second variation.

In the Lagrangian mechanical description, a stationary solution is expressed by the exponential map of  $V$ , i.e., the reference path is given by  $\gamma(t, 0) = e^{tV}$ . Then, using the displacement field,  $\xi(t)$ , associated with the perturbed solution  $V_{\epsilon}(t) = V + \epsilon v(t)$ , the perturbed path,  $\gamma(t, \epsilon)$ , induced by  $V_{\epsilon}(t)$  is expressed as  $\gamma(t, \epsilon) = e^{\xi(t)} e^{tV}$ . Furthermore, the path that is obtained by parallel translation of the path  $\gamma(t, \epsilon)$  by  $\tau$  along  $\gamma(t, 0) = e^{tV}$ , i.e., the path  $\gamma'(t, \epsilon) := e^{\xi(t)} e^{(t+\tau)V}$  can also be realized. The schematic view is given in Fig. 2. Hence, the value of Eq. (25) is unchanged and the related constant of motion may exist.

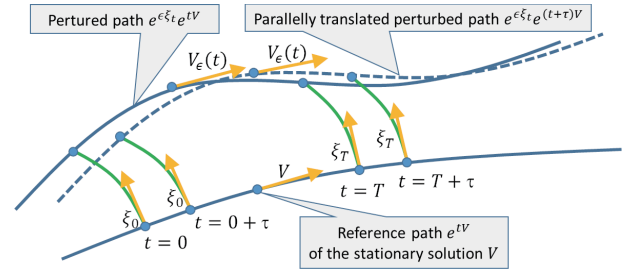


Fig. 2 Schematic view of the parallel translation of the perturbed path.

## 4. Comparison with Hamiltonian Mechanical Analysis

The stability of plasma configuration in some MHD systems has been frequently analyzed in the Hamiltonian mechanical framework. One of the well-known analysis methods is the energy-Casimir method (e.g. [2]; Section 1.7). An important improvement of the variational procedure was made by Hirota *et al.* [8]. The authors distinguished between the following three kinds of perturbation fields: arbitrary perturbations, Lagrangian displacements (LD), and dynamically accessible variations (DAV).

### 4.1 Derivation of the second variation of the Hamiltonian

While the LD approach is based on relation (8), the DAV approach is based on the Poisson bracket that is defined by

$$\{F, G\}(\underline{M}) := \left( \underline{M} \left| \left[ \frac{\delta F}{\delta \underline{M}}, \frac{\delta G}{\delta \underline{M}} \right] \right. \right), \quad (26)$$

where  $\underline{M}$  is generalized momentum,  $F$  and  $G$  are functionals of generalized momentum, and  $(\underline{M} | V)$  is inner product of the generalized momentum and velocity. The DAV of a certain physical quantity  $F$  induced by the perturbation  $K$  is given by their Poisson bracket,  $\delta F_{da} = \{F, K\}$ .

The first variation of the Hamiltonian  $H(\underline{M}) := \frac{1}{2} (\underline{M} | V) = \frac{1}{2} \langle V | V \rangle$  against the perturbation  $K$  is

$$\{H, K\}(\underline{M}) = (\underline{M} | [V, \xi]) = \langle V | L_{\xi} V \rangle = -\langle L_V^{\dagger} V | \xi \rangle, \quad (27)$$

where  $\frac{\delta H}{\delta \underline{M}} = V$ ,  $\frac{\delta K}{\delta \underline{M}} = \xi$ . Note that, when the reference solution is stationary,  $\dot{V} = L_V^{\dagger} V = 0$ , the first variation (27) vanishes, which implies the stationary solution is an extremal of the Hamiltonian.

The value of the functional derivative of  $\{H, K\}$  around  $\underline{M}$  is obtained by substituting  $\underline{M} + \epsilon \underline{m}$  into Eq. (27):

$$\begin{aligned} \left( \underline{m} \left| \frac{\delta \{H, K\}}{\delta \underline{M}} \right. \right) &:= \frac{d}{d\epsilon} \{H, K\}(\underline{M} + \epsilon \underline{m}) \Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \langle L_{\xi}^{\dagger} (V + \epsilon v) | V + \epsilon v \rangle \Big|_{\epsilon=0} \\ &= \langle L_{\xi}^{\dagger} v | V \rangle + \langle L_{\xi}^{\dagger} V | v \rangle = \langle L_{\xi} V + L_{\xi}^{\dagger} v | v \rangle \\ &= (\underline{m} | L_{\xi} V + L_{\xi}^{\dagger} V), \end{aligned} \quad (28)$$

where  $v$  is the dual of  $\underline{m}$ , (i.e.,  $(\underline{m} | \xi) = \langle v | \xi \rangle$  for an arbitrary  $\xi$ ). Using this relation, we obtain the second variation of

<sup>7</sup>Note that  $\frac{d}{dt} \langle V | v \rangle = 0$  for arbitrary solution of Eqs. (13) and (16).

the Hamiltonian as follows:

$$\begin{aligned}\delta_{DAV}^2 H &= \{H, K\}, K\}(\underline{M}) = \{Q, K\}(\underline{M}) = \left(\underline{M} \left[ \frac{\delta Q}{\delta \underline{M}}, \frac{\delta K}{\delta \underline{M}} \right]\right) \\ &= \left(\underline{M} \left[ L_\xi \frac{\delta Q}{\delta \underline{M}} \right]\right) = \langle V | L_\xi \frac{\delta Q}{\delta \underline{M}} \rangle = \langle L_\xi^\dagger V | \frac{\delta Q}{\delta \underline{M}} \rangle \\ &= \langle L_\xi^\dagger V | L_\xi^\dagger V + L_\xi V \rangle = \langle L_\xi^\dagger V | L_\xi^\dagger V \rangle + \langle L_\xi V | L_\xi^\dagger V \rangle,\end{aligned}\quad (29)$$

where  $Q = \{H, K\}$ .

## 4.2 Stationary solutions: the double Beltrami flow and its MHD counterpart

It is shown in Ref. [5] that the Lagrangian mechanical analysis of the dynamics of the incompressible fluids and plasmas naturally derives the notion of *generalized vorticity* as well as generalized velocity and momentum. Generalized vorticity is shown to be obtained by operating the helicity-based particle-relabeling operator on generalized velocity. One of the important finding is the eigenfunctions of the operator provide stationary force-free solutions of the corresponding system.

In this study the pair of the velocity and current fields is taken as the generalized velocity ( $\vec{V} := (V, -\alpha J)$ ), where  $\alpha$  is the Hall term strength parameter for the HMHD system and an appropriate constant for the MHD system. The helicity-based particle-relabeling operators are given by

$$\widehat{W}_{HMHD} = \begin{pmatrix} C_C \alpha \nabla \times & -C_C (\alpha \nabla \times)^{-1} \\ -C_C \alpha \nabla \times & C_M (\alpha \nabla \times)^{-1} \end{pmatrix}, \quad (30)$$

for the ideal incompressible HMHD system, and

$$\widehat{W}_{MHD} = \begin{pmatrix} O & -C_C (\alpha \nabla \times)^{-1} \\ -C_C \alpha \nabla \times & C_M (\alpha \nabla \times)^{-1} \end{pmatrix}, \quad (31)$$

for the ideal incompressible MHD system, where  $C_C$  and  $C_M$  are arbitrary constants related to the helicity conservation laws associated with the particle-relabeling operation. When  $C_C \neq 0$ , the corresponding eigenvalue and eigenequations are given by

$$\begin{aligned}\Lambda &= \Lambda_\sigma^s(\lambda) \\ &= \sigma s C_C \left[ \sqrt{1 + \left( \frac{\alpha \lambda}{2} - \frac{\tilde{C}_M}{2\alpha \lambda} \right)^2} + s \left( \frac{\alpha \lambda}{2} + \frac{\tilde{C}_M}{2\alpha \lambda} \right) \right],\end{aligned}\quad (32)$$

$$\alpha \Omega + B = \frac{\Lambda}{C_C} V, \quad B = \frac{\Lambda}{C_C(1 - \tilde{C}_M)} (V - \alpha J), \quad (33)$$

for the HMHD system, and

$$\Lambda = \Lambda_\sigma^s(\lambda) = \sigma s C_C \left[ \sqrt{1 + \left( \frac{\tilde{C}_M}{2\alpha \lambda} \right)^2} + \frac{s \tilde{C}_M}{2\alpha \lambda} \right], \quad (34)$$

$$\Omega = -\frac{\tilde{C}_M \Lambda}{C_C \alpha} V + \frac{\Lambda}{C_C} J, \quad B = \frac{\Lambda}{C_C} V, \quad (35)$$

for the MHD system, where  $\sigma = \pm 1$ ,  $s = \pm 1$ ,  $\Omega = \nabla \times V$ ,  $B = (\nabla \times)^{-1} J$ ,  $\tilde{C}_M := C_M/C_C$ , and  $\lambda$  and  $\Lambda$  are the eigenvalues of  $\nabla \times$  and  $\widehat{W}$ , respectively. Figure 3 shows the

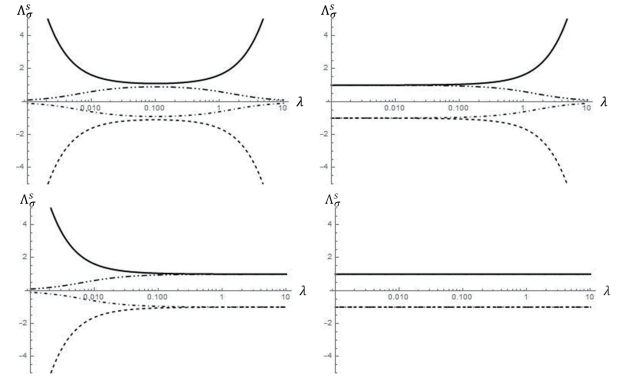


Fig. 3 Typical  $\lambda$  dependence of  $\Lambda_\sigma^s(\lambda)$  for each  $\sigma$  and  $s$ . Top: the HMHD case (Eq. (32)), bottom: the MHD case (Eq. (34)). Left:  $\tilde{C}_M = 0.01$ , right:  $\tilde{C}_M = 0$ . Solid:  $\Lambda_+^s$ , dashed-two dotted:  $\Lambda_-^s$ , dashed-dotted:  $\Lambda_+^-$ , dashed:  $\Lambda_-^-$ . Abscissas are plotted in logarithmic scale for  $\lambda \in [0.001, 10]$ .

$\lambda$  dependence of the eigenvalues  $\Lambda$ . Generally, if  $C_C > 0$  and  $C_M \geq 0$ , the eigenvalues satisfy

$$\Lambda_-^+ \leq -1 \leq \Lambda_+^- < 0 < \Lambda_-^- \leq 1 \leq \Lambda_+^s,$$

for both the HMHD and MHD cases.

The eigenfunction of Eq. (33) is known as the *double Beltrami flow* (DBF) [9]:

$$\vec{\Phi}_\sigma^s(\lambda) = \begin{pmatrix} V_\sigma^s(\lambda) \\ -\alpha J_\sigma^s(\lambda) \end{pmatrix} = \left( \left( \Lambda - \frac{C_M}{\alpha \lambda} \right) \phi, -C_C \alpha \lambda \phi \right), \quad (36)$$

where  $\phi$  is the eigenfunction of  $\nabla \times$  with eigenvalue  $\lambda$ . Note that the eigenequation and corresponding eigenfunctions for the MHD system are also obtained as the  $\alpha \rightarrow 0$  limit of that of the HMHD system (if  $\tilde{C}_M \approx O(\alpha)$ ).

When  $C_C = 0$ , on the other hand, the equilibrium solution only has a current field component and is given by a Beltrami function:  $(V, -\alpha J) = (0, -\alpha \lambda B)$ .

## 4.3 Stability analyses in MHD and HMHD systems

Setting the generalized velocity and the associated displacement field as  $\vec{V} = (V, -\alpha J)$ ,  $\vec{\xi} = (\xi, -\alpha \eta)$ , and using the notations  $U := V - \alpha J$ ,  $\zeta := \xi - \alpha \eta$ , the inner product, the Lie bracket, and 2nd-variation-related integrals are expressed as follows:

$$\langle \vec{V}_1 | \vec{V}_2 \rangle = \int (V_1 \cdot V_2 + B_1 \cdot B_2) d^3 x, \quad (37)$$

$$[\vec{V}_2, \vec{V}_3] = (\nabla \times (V_2 \times V_3), \nabla \times (U_2 \times U_3 - V_2 \times V_3)), \quad (38)$$

$$\langle L_\xi \vec{V} | L_\xi^\dagger \vec{V} \rangle = \int \{ |\nabla \times (V \times \xi)|^2 + \alpha^{-2} |V \times \xi - U \times \zeta|^2 \} d^3 x, \quad (39)$$

$$\begin{aligned}\langle L_\xi \vec{V} | L_\xi^\dagger \vec{V} \rangle &= \frac{1}{\alpha} \int \{ [\nabla \times (V \times \xi)] \cdot [\xi \times (\alpha \Omega + B) - \zeta \times B] \\ &\quad + (V \times \xi - U \times \zeta) \cdot [\nabla \times (\zeta \times B)] \} d^3 x,\end{aligned}\quad (40)$$

$$\langle L_{\xi}^{\dagger} \vec{V} | L_{\xi}^{\dagger} \vec{V} \rangle = \int \left\{ \alpha^{-2} |\xi \times (\alpha \Omega + \mathbf{B}) - \xi \times \mathbf{B}|^2 + |\nabla \times (\xi \times \mathbf{B})|^2 \right\} d^3 \vec{x}. \quad (41)$$

Assuming that the variables are single-mode DBFs,  $\vec{V} = \widehat{V}_{\tilde{p}} \vec{\Phi}_{\tilde{p}}$  and  $\vec{\xi} = \widehat{\xi}_{\tilde{k}} \vec{\Phi}_{\tilde{k}}$  where  $\tilde{p} = (\sigma_p, s_p, \lambda_p)$  and  $\tilde{k} = (\sigma_k, s_k, \lambda_k)$  are the appropriate mode indices, and using the relation ,

$$\langle \vec{\Phi}_{\tilde{q}} | [\vec{\Phi}_{\tilde{p}}, \vec{\Phi}_{\tilde{k}}] \rangle = \Lambda_{\tilde{q}} T_{\tilde{q}\tilde{p}\tilde{k}},$$

or equivalently

$$[\vec{\Phi}_{\tilde{p}}, \vec{\Phi}_{\tilde{k}}] = \sum_{\tilde{q}} g_{\tilde{q}}^{-1} \Lambda_{\tilde{q}} T_{\tilde{q}\tilde{p}\tilde{k}} \vec{\Phi}_{\tilde{q}}^*,$$

where  $T_{\tilde{q}\tilde{p}\tilde{k}}$  is a totally antisymmetric tensor and  $g_{\tilde{q}} := \langle \vec{\Phi}_{\tilde{q}} | \vec{\Phi}_{\tilde{q}}^* \rangle$ , we obtain

$$\langle L_{\xi}^{\dagger} \vec{V} | L_{\xi}^{\dagger} \vec{V} \rangle = |\widehat{V}_{\tilde{p}}|^2 |\widehat{\xi}_{\tilde{k}}|^2 \sum_{\tilde{q}} \frac{\Lambda_{\tilde{q}}^2}{g_{\tilde{q}}} |T_{\tilde{q}\tilde{p}\tilde{k}}|^2, \quad (42)$$

$$\langle L_{\xi}^{\dagger} \vec{V} | L_{\xi}^{\dagger} \vec{V} \rangle = -\Lambda_{\tilde{p}} |\widehat{V}_{\tilde{p}}|^2 |\widehat{\xi}_{\tilde{k}}|^2 \sum_{\tilde{q}} \frac{\Lambda_{\tilde{q}}}{g_{\tilde{q}}} |T_{\tilde{q}\tilde{p}\tilde{k}}|^2, \quad (43)$$

$$\langle L_{\xi}^{\dagger} \vec{V} | L_{\xi}^{\dagger} \vec{V} \rangle = \Lambda_{\tilde{p}}^2 |\widehat{V}_{\tilde{p}}|^2 |\widehat{\xi}_{\tilde{k}}|^2 \sum_{\tilde{q}} \frac{1}{g_{\tilde{q}}} |T_{\tilde{q}\tilde{p}\tilde{k}}|^2. \quad (44)$$

Thus, the second variations become

$$I = |\partial_t \widehat{\xi}_{\tilde{k}}|^2 + |\widehat{V}_{\tilde{p}}|^2 |\widehat{\xi}_{\tilde{k}}|^2 \sum_{\tilde{q}} \frac{\Lambda_{\tilde{q}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}})}{g_{\tilde{q}}} |T_{\tilde{q}\tilde{p}\tilde{k}}|^2, \quad (45)$$

$$\delta_{DAV} H = |\widehat{V}_{\tilde{p}}|^2 |\widehat{\xi}_{\tilde{k}}|^2 \sum_{\tilde{q}} \frac{\Lambda_{\tilde{p}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}})}{g_{\tilde{q}}} |T_{\tilde{q}\tilde{p}\tilde{k}}|^2. \quad (46)$$

The expressions formally agree with those found in the Section 5.3 of Ref. [4], where the results were limited to the HMHD system and the mode expansion by the generalized Elsässer variables. According to our consideration made in Ref. [5], these expressions are now applicable to wider classes of dynamical systems including the MHD and HMHD systems.

Let us consider the stability problem of the stationary solutions of the MHD and HMHD systems using these two expressions. The conservation law given by Eq. (45) tells us that, if the second term is positive definite, the norm of  $|\partial_t \widehat{\xi}_{\tilde{k}}|^2$  and  $|\widehat{\xi}_{\tilde{k}}|^2$  are bounded, and thus, the growth of the Lagrangian displacement field  $\vec{\xi}$  is limited and the solution  $\vec{V}$  is Lyapunov stable. On the other hand, the second variation of the Hamiltonian around a stationary solution (Eq. (46)) has different meaning. If the second variation is positive or negative definite, the Hamiltonian takes an extremal value at the stationary solution, which implies that the norm of the deviation from the stationary solution is bounded, (i.e., the perturbed solution is Lyapunov stable). In this context, the kind of perturbations is somewhat restricted to LD or DAV perturbations, and their relation to well-known instabilities such as kink or Kelvin-Helmholtz requires more careful consideration.

For the MHD case, as is seen from the bottom two plates of Fig. 3, if  $\lambda_0$  is the smallest eigenvalue of  $\nabla \times$ , the eigenvalues given by Eq. (34) satisfy

$$\Lambda_{\tilde{p}}^+(\lambda_0) \leq \Lambda_{\tilde{p}}^+ \leq -1 \leq \Lambda_{\tilde{p}}^- < 0 < \Lambda_{\tilde{p}}^- \leq 1 \leq \Lambda_{\tilde{p}}^+ \leq \Lambda_{\tilde{p}}^+(\lambda_0).$$

Especially, if  $\tilde{C}_M = 0$ , the eigenvalues satisfy  $\Lambda_{\tilde{p}}^+ = \Lambda_{\tilde{p}}^- = -1$ ,  $\Lambda_{\tilde{p}}^- = \Lambda_{\tilde{p}}^+ = 1$ . In these cases, the coefficients satisfy  $\Lambda_{\tilde{p}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}}) \geq 0$  for all  $\tilde{q}$ , and thus, the second variation of the Hamiltonian is positive definite ( $\delta_{DAV} H \geq 0$ ), implying that the solutions corresponding to  $\Lambda_{\tilde{p}}^-(\lambda_0)$  and  $\Lambda_{\tilde{p}}^+(\lambda_0)$  are Lyapunov stable.

While the stability conditions related to the constant  $I$ , (i.e.,  $\Lambda_{\tilde{q}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}}) \geq 0$  for all  $\Lambda_{\tilde{q}}$ ),<sup>8</sup> are impossible for the MHD case. Thus, we cannot draw a definite conclusion from the analysis of Eq. (45).

Conversely, for the HMHD case both of the coefficients of Eqs. (45) and (46),  $\Lambda_{\tilde{q}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}})$  and  $\Lambda_{\tilde{p}}(\Lambda_{\tilde{p}} - \Lambda_{\tilde{q}})$ , can assume both positive and negative values, and thus, no definitive information is obtained from the present analysis. This indefiniteness is caused by the Hall term effect. The eigenvalues,  $\Lambda_{\tilde{q}}$ , which are firmly related to the phase velocity of the whistler and ion-cyclotron waves [5], diverges for whistler wave branches and asymptote to zero for the ion-cyclotron branches.

In this relation, we should mention the results of Hirota *et al.* [8], in which the authors carefully rearrange terms to figure out the definiteness of the second variation of the Hamiltonian was carried out. Compared with their method, our method seems relatively crude for evaluating the stability criterion. However, our method can treat the stability problems of the MHD and HMHD systems in a unified manner.

## Acknowledgments

This study was performed under the auspices of the NIFS Collaboration Research Program (NIFS18KNSS107) and KAKENHI (Grant-in-Aid for Scientific Research(C)) 17K05734.

- [1] V.I. Arnold, *Mathematical methods of classical mechanics* 2nd Ed. (Springer, New York, 1989).
- [2] J.E. Marsden and T. Ratiu, *Introduction to mechanics and symmetry* (Springer, New York, 1999).
- [3] Y. Hattori, J. Phys. A: Math. Gen. **27**, L21 (1994).
- [4] K. Araki, J. Phys. A: Math. Theor. **50**, 235501 (2017).
- [5] K. Araki, submitted to Phys. Plasmas; arXiv:1601.05477v2 (2018).
- [6] S. Kobayashi and K. Nomizu, *Foundations of differential geometry. Volume I* (Interscience, New-York, 1963).
- [7] E. Noether, Gott. Nachr. **1918**, 235 (1918); transl. M.A. Tavel, arXiv:physics/0503066.
- [8] M. Hirota, Z. Yoshida and E. Hameiri, Phys. Plasmas **13**, 022107 (2006).
- [9] S.M. Mahajan and Z. Yoshida, Phys. Rev. Lett. **81**, 4863 (1998).

<sup>8</sup>In other words, all the  $\Lambda_{\tilde{q}}$  satisfies  $0 \leq \Lambda_{\tilde{q}} \leq \Lambda_{\tilde{p}}$  or  $\Lambda_{\tilde{p}} \leq \Lambda_{\tilde{q}} \leq 0$  for an assigned DBF  $\vec{\Phi}_{\tilde{p}}$ .