High-Pressure Limit of Equilibrium in Axisymmetric Open Traps*)

Mikhail S. KHRISTO and Alexey D. BEKLEMISHEV

Budker Institute of Nuclear Physics SB RAS, Novosibirsk 630090, Russia
Novosibirsk State University, Novosibirsk 630090, Russia

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The high pressure limit of equilibrium in linear traps corresponds to the diamagnetic reduction of the confining field and the corresponding increase in the volume of flux tubes. In the gas-dynamic regime the axial losses from a flux tube are proportional to its cross-section in the mirror throat. Thus, the axial confinement time, which is proportional to the ratio of the flux-tube volume to its cross-section in the mirror throat, can grow significantly in the high-pressure limit. In this paper the numerical model of the axially symmetric equilibrium based on the coupled Grad-Shafranov and transport equations is presented. The results are in good agreement with the earlier analytical model [A.D. Beklemishev *et al.*, Fusion Sci. Technol. **63**(1T), 46 (2013)].

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1. Introduction

Linear systems, such as open traps and field-reversed configurations (FRC), become more and more effective, and the use of open trap as a compact fusion reactor with a high energy density and a relatively simple design is promising. The gas-dynamic trap is one of the candidates for this role. However, theoretical estimates show that in order to achieve fusion parameters in gas-dynamic trap an extremely long system is required [1].

The recently proposed diamagnetic confinement mode at high beta [1] allows the significant reduction in reactor length due to the improved confinement. The diamagnetic trap is considered as a possible part of the GDMT program [2].

If the plasma is pumped into a magnetic tube faster than it flows out through the ends, the cross section of the tube increases due to the diamagnetic weakening of the field at the constant magnetic flux. The increase in cross-section occurs mainly in the region of a weak vacuum magnetic field. The larger cross-section leads to the larger tube volume V, while the end-loss remains almost constant. As a result, the longitudinal particle lifetime in the flux-tube increases, since it is proportional to the plasma volume, indeed

$$\tau_{\parallel} \sim \frac{Vn}{2 \times nv_m \times S_m} \sim \frac{LR_{vac}}{2v_m}\frac{1}{\sqrt{1-\beta}},$$

where n is the plasma density, S_m and v_m are the flux-tube cross-sectional area and the flow velocity in the mirror throat respectively, L is the distance between the mirrors, R_{vac} is the vacuum mirror ratio, $\beta = 8\pi p/B_0^2$ is the relative plasma pressure, B_0 is the magnitude of the vacuum magnetic field in the center of the trap. Thus, in the high-pressure limit ($\beta \to 1$), the longitudinal particle lifetime in the magnetic tube formally tends to infinity ($\tau_{\parallel} \to \infty$).

As it was shown in Ref. [1], if one creates a patch of quasi-homogeneous magnetic field in the vacuum magnetic configuration, the cylindrical diamagnetic "bubble" is formed in this region. Inside the "bubble", the transverse flow is much greater than the axial loss, since the magnetic field is extremely small there – it is almost completely displaced by the plasma. There is no plasma outside due to the large axial losses. Consequently, the transition layer – the region of the sharp plasma pressure and the magnetic field gradient – is formed at the "bubble" boundary. The thickness of the transition layer λ can be estimated from the flow continuity condition:

$$\Phi_{\perp} \sim D\frac{n}{\lambda} \times 2\pi aL, \quad \Phi_{\parallel} \sim 2 \times nv_m \times \frac{2\pi a\lambda}{R_{vac}},$$

$$\Phi_{\perp} \sim \Phi_{\parallel} \quad \Rightarrow \quad \lambda \sim \sqrt{\frac{D_M LR_{vac}}{2v_m}},$$

where $\Phi_{\perp}$, $\Phi_{\parallel}$ are the transverse and longitudinal plasma flows respectively, $D_M = c^2/4\pi\sigma$ is the diffusion coefficient across the magnetic field, a is the "bubble" radius. In the same way, the particle lifetime in the "bubble" can be estimated

$$\tau_n \sim \frac{Vn}{2 \times nv_m \times 2\pi a\lambda/R_{vac}} \sim \tau_{GDT}\frac{a}{\lambda} \sim \sqrt{\tau_{\perp}\tau_{GDT}},$$

where $\tau_{GDT} = LR_{vac}/2v_m$ is the gas-dynamic lifetime, $\tau_{\perp} = a^2/D_M$ is the diffusion time across the magnetic field. For fusion parameters, the transition layer λ is extremely thin, which means that the particle lifetime in the "bubble" increases significantly:

$$\tau_n \sim \tau_{GDT}\frac{a}{\lambda} \sim \sqrt{\tau_{\perp}\tau_{GDT}} \gg \tau_{GDT}.$$

author's e-mail: khristo.mikhail@gmail.com

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Analytical equilibrium-transport model of plasma in the diamagnetic confinement mode using the cylindrical approximation was presented in Ref. [1]. However, one conclusion of paper [1] was that in the high-beta limit of plasma equilibrium there will appear non-paraxial areas, so that the analytical theory has limited applicability. Thus, the need of building more accurate model arose.

In this work we used the simple theoretical model of plasma equilibrium and transport, consisting of the Grad-Shafranov equation [3, 4] and the particle transport equation, while the temperature is considered constant. Strong nonlinearity of the equilibrium-transport equations in the non-paraxial case requires application of numerical methods, in particular, the finite difference method and the iterative method were used in present paper.

2. Theoretical Model

2.1 Equilibrium

To describe the equilibrium, we apply the Grad-Shafranov equation that is often used for axisymmetric systems. The derivation of this equation in general case is given in Ref. [3,4]. In the same way, the equation can be obtained for the case of the axisymmetric open trap:

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\psi}{\partial r}\right)+\frac{\partial^2\psi}{\partial z^2}=-16\pi^3 r^2\frac{dp}{d\psi}-\frac{8\pi^2}{c}r j_{vac}, \quad (1)$$

where j_{vac} is the current density in the external conductors, p is the plasma pressure, ψ is the magnetic flux through the $2\pi r dr$ cross section, which in cylindrical coordinates (r, θ, z) is defined as

$$[\mathbf{B}\times\mathbf{e}_\theta]=-\frac{1}{2\pi r}\nabla\psi. \quad (2)$$

It is often assumed that the plasma is enclosed in an ideally conducting shell. In the general case, when the conducting shell does not coincide with the magnetic surface, for the time intervals of the order of the skin-effect time the boundary condition can be formulated as follows

$$\psi(r,z)=\psi_0(r,z), \qquad (r,z)\in\gamma_s.$$

where γ_s is the curve in (r, z) space, which coincides with the conducting shell, $\psi_0(r, z)$ is the given function determined by the external fields and the shell shape.

In present paper it is assumed that there is no conducting shell around the plasma. In this case, the boundary conditions are set as follows

$$\psi(r,z)|_{r=0}=0, \qquad \lim_{r,z\to\infty}|\psi(r,z)|\leqslant \text{const}<\infty.$$

The simplified method of setting boundary condition for z is the periodicity of the solution:

$$\psi(r,z)=\psi(r,z+L).$$

In this case, the discrete Fourier transform is applicable, which greatly simplifies the numerical simulation. The solutions for a trap of finite length and periodic one are close if the distance from the mirror to the center of the trap is much greater than the radius of the plasma, i.e. $L/2 \gg a$.

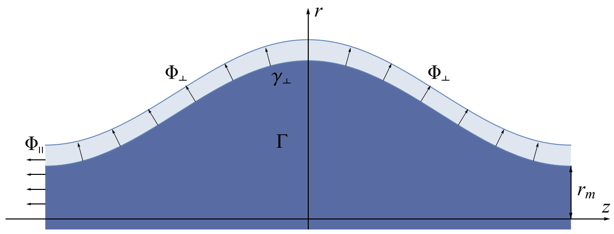

Fig. 1 The cross-section of the magnetic tube in the (r, z) plane.

2.2 Transport

Right-hand side of the Grad-Shafranov Eq. (1) contains the pressure profile $p = p(\psi)$. This function is often specified by some model one, for example, found from the approximation of experimental data. However, the diamagnetic confinement mode equilibrium is realized at the limiting pressure.

The limiting pressure profile can be determined from the transport equation, which can be derived from the law of particle number conservation:

$$\frac{d}{d\psi}\left(\Phi_\parallel+\Phi_\perp\right)=\frac{dQ}{d\psi}, \quad (3)$$

where $\Phi_\parallel$ and $\Phi_\perp$ are longitudinal and transverse flows of matter respectively, Q is the internal particle source (Fig. 1).

The flux of the plasma across the magnetic field is determined by the diffusion of the magnetic field through the plasma. Thus, the transverse flow velocity is determined by the conductivity of the plasma and can be found from Ohm's law:

$$\frac{1}{c}[\mathbf{v}\times\mathbf{B}]+\mathbf{E}=\frac{\mathbf{j}}{\sigma}.$$

It can be shown that the azimuthal component of the electric field equals to zero. Substituting the magnetic field from Eq. (2) one can obtain

$$\mathbf{v}_\perp=-\frac{2\pi rc}{\sigma}j_\theta\frac{\nabla\psi}{|\nabla\psi|^2}. \quad (4)$$

In turn, the force balance

$$\nabla p=\frac{1}{c}\left[\mathbf{j}\times\mathbf{B}\right],$$

together with Eq. (2), yields

$$j_\theta=2\pi rc\frac{dp}{d\psi}. \quad (5)$$

Eventually, substituting the azimuth current (5) into the expression (4), we obtain

$$\Phi_\perp=\int\limits_{S_\perp(\psi)}n\mathbf{v}d\mathbf{S}=-\frac{4\pi^2c^2}{\sigma}n\frac{dp}{d\psi}\int\limits_{\gamma_\perp(\psi)}\frac{r^2dz}{B_z}, \quad (6)$$

where the integration is carried out along the field line ψ = const (Fig. 1), B_z is the z-component of the magnetic field.

The expression for the longitudinal flow through the ends of the flux-tube has the form

$$\Phi_{\parallel} = 2 \int\limits_{S_{\parallel}(\psi)} n\mathbf{v}d\mathbf{S} = 4\pi \int\limits_{0}^{r_m(\psi)} n v_m r dr, \qquad (7)$$

where r_m, v_m are the flux-tube radius and the flow velocity in the mirror throat respectively.

For simplicity, we assume the electron temperature to be constant T = const, and hence σ = const and v_m = const. Then the equation of state $p = nT$, together with the expressions (6, 7), leads to the transpot equation

$$2v_m \frac{p}{B_{mz}} - (2\pi)^3 D_M \frac{d}{d\psi}\left(\frac{dp^2}{d\psi} \int\limits_{\gamma_\perp(\psi)} \frac{r^2 dz}{B_z}\right) = \frac{dW}{d\psi}, \quad (8)$$

where $W = TQ$ is the energy source, B_{mz} is the value of B_z in the mirror.

It can be assumed that the plasma is bounded by the limiter of radius a_{lim}. Then, the plasma pressure beyond the limiter is zero, which can formally be represented as

$$p(r \geqslant a_{lim}) = 0.$$

Note that the limiter can be cut into sections to avoid inducing azimuth current in it.

The boundary condition on the axis can be set as follows

$$\left.\frac{\partial p}{\partial r}\right|_{r=0} = 0.$$

If the particle source is localized near the axis and its radius is less than a_s, this condition can be replaced by the approximate one. Neglecting the longitudinal flow term, we have

$$\left.\frac{dp^2}{d\psi}\right|_{r=a_s} \approx -\left. W \left((2\pi)^3 D_M \int\limits_{\gamma_\perp(\psi)} \frac{r^2 dz}{B_z}\right)^{-1}\right|_{r=a_s}.$$

This statement of the boundary condition can be especially useful for numerical calculations, if the source size is comparable with the step of a discrete grid.

3. Numerical Simulation

Due to the fact that the system of the equilibrium-transport Eqs. (1, 8) is significantly nonlinear, the application of numerical methods is required to solve it. In particular, the finite difference method and iteration method are used in present paper.

3.1 Diamagnetic "bubble" equilibrium

First, let us consider solution that describes the equilibrium of the plasma in the regime of diamagnetic confinement. The equilibrium distribution of the magnetic field obtained in numerical calculations and the corresponding

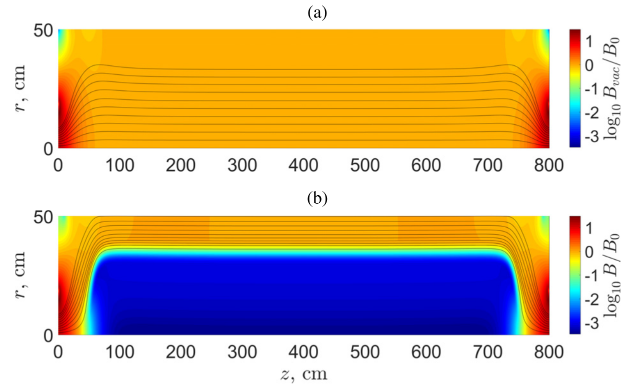

Fig. 2 Distribution of the magnetic field in the (r, z) plane. (a): vacuum magnetic configuration; (b): field with the plasma. Black lines denote magnetic field lines.

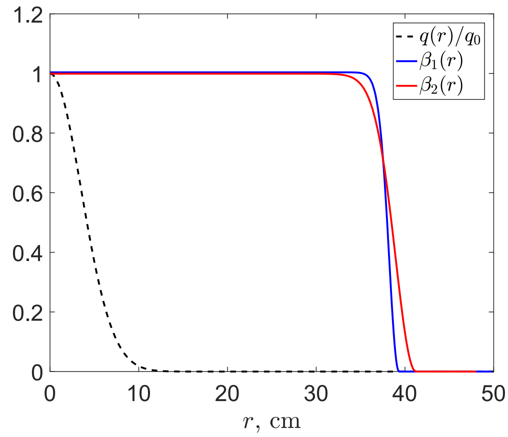

Fig. 3 Relative plasma pressure $\beta(r) = 8\pi p(r)/B_0^2$ radial profiles (red and blue) and the radial shape of the particle source $q(r)$ (black dashed) in the central section of the trap.

vacuum configuration are shown in Fig. 2. Figure 3 represents the radial distribution of relative plasma pressure $\beta(r) = 8\pi p(r)/B_0^2$, where B_0 is the magnitude of the vacuum magnetic field in the center of the trap.

As it was expected, in the center of the "bubble", the magnetic field is almost completely displaced by the plasma and close to zero ($B \sim 10^{-3} B_0$), and the plasma β in turn is close to unity.

Analytical estimates of the transition layer thickness and the "bubble" radius were obtained in Ref. [1].

$$\lambda \sim \sqrt{\frac{D_M L R_{vac}}{2v_m}}, \qquad a \sim \frac{Q}{2\pi D_M n/\lambda}. \qquad (9)$$

To verify the correctness of the numerical solutions, we investigated its scaling characteristics. For this purpose, numerical solutions for different relationship between the transport parameters D_M and v_m were constructed. In particular, Fig. 3 shows the pressure profiles for two numerical solutions constructed for the same vacuum magnetic field configuration and different ratio between the transport parameters. The second $\beta_2(r)$ differs from the first

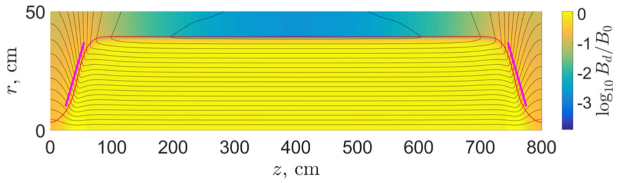

Fig. 4 Distribution of the diamagnetic field in the (r, z) plane. Red color indicates the magnetic surface corresponding to the plasma boundary ($\beta \sim 0.01$). The preliminary position of the shell stabilization plates is indicated in purple.

$\beta_1(r)$ in that it was calculated at double the transverse transport coefficient ($D_{M2} = 2D_{M1}$) and half the longitudinal one ($v_{m2} = v_{m1}/2$). In turn, according to the above estimates (9), the radius of the "bubble" should be the same for both solutions ($a_2 = a_1$), and the transition layer in the second case should be twice as thick as in the first case ($\lambda_2 = 2\lambda_1$). It can be seen that the numerical solutions are in good agreement with analytical scaling.

3.2 Diamagnetic field

One of the methods considered for stabilizing the plasma in the diamagnetic trap is the stabilization by the conducting shell. In Ref. [5], in the case of an axisymmetric open trap, it was shown that the region of robust shell stabilization exists for the plasma with hot ions and β exceeding some threshold value [6]. Therefore, the optimization of the shell stabilization plates position is the relevant issue. The distribution of the diamagnetic field was also calculated for this purpose. Plates should be installed in the region of the most intense diamagnetic field close enough to the plasma boundary. Preliminary position of the shell stabilization plates is shown in Fig. 4. To prevent the induction of azimuth current, the plates can be cut into segments, as stabilization is carried out by Foucault eddy currents.

3.3 Magnetic field corrugation

In the experiment, the vacuum magnetic field is inevitably modulated, since it is formed by discretely located coils. In this connection, the study of the influence of vacuum magnetic field corrugation on the diamagnetic "bubble" equilibrium can be useful. An example of the equilibrium solution in a corrugated field is shown in Fig. 5.

Based on the results of a set of numerical experiments carried out for various corrugation parameters such as step h and amplitude δB, the dependence shown in the Fig. 6 (a) was constructed. In addition, the analytical approximation of this dependence was obtained (Fig. 6 (b)):

$$\frac{\delta r}{r} = \frac{\delta B}{B} \mathrm{H}\left(\frac{2\pi r}{h}\right), \quad \mathrm{H}(x) = \frac{1}{x}\left[\frac{I_1(x)}{I_0(x)} + \frac{K_1(x)}{K_0(x)}\right].$$

As one can see from the diagrams, the numerical and analytical results are consistent at least at a qualitative level.

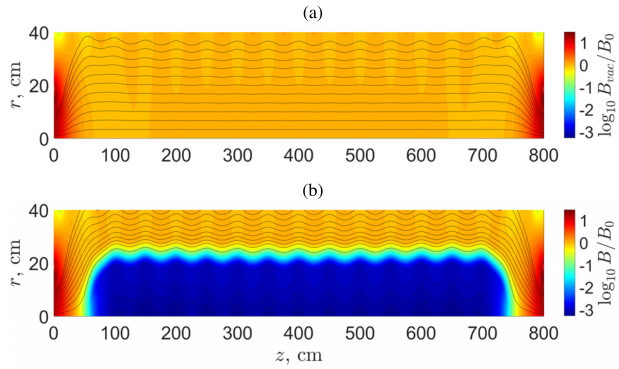

Fig. 5 Distribution of the corrugated magnetic field in the (r, z) plane. (a): vacuum magnetic configuration; (b): field with the plasma.

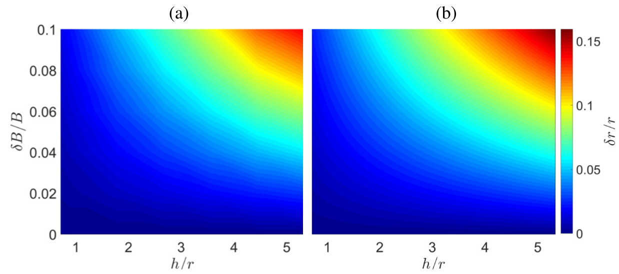

Fig. 6 Dependence of the magnetic lines corrugation at the "bubble" boundary $\delta r/r$ on the corrugation of the vacuum magnetic field $\delta B/B$ and corrugation step h/r. (a): modeling; (b): analytical approximation.

4. Conclusion

The MHD model for calculating axisymmetric stationary plasma equilibria in open traps was developed. Numerical equilibria corresponding to the diamagnetic confinement mode are constructed. They are in good agreement the with analytical estimates based on Ref. [1]. It is possible to optimize the position of the shell stabilization plates using the calculations of the diamagnetic field.

The influence of the vacuum field corrugation on the diamagnetic confinement mode equilibrium was investigated. Both numerical and analytical results were obtained. One can conclude that the corrugation of the vacuum magnetic field leads to a proportional corrugation of the "bubble" boundary. This allows the formulation of requirements on amplitude of ripple fields in GDMT [2].

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