Lyapunov Exponent Analysis for Stochastic Ion Motion in a Non-Adiabatic Trap[*)]

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The Lyapunov exponent analysis is carried out for ion motions in a non-adiabatic trap where the axial magnetic field from a solenoid is partially cancelled by that from a Helmholtz coil. In particular, relation between the Lyapunov exponents and the trapped rate of ions injected in the axial direction is numerically studied. It is found that there exist energy minimizing the particle trapped rate, and the Lyapunov exponent for the axial velocity is found to be minimized with that energy.

Keywords: non-adiabatic trap, Helmholtz coil, stochastic motion, particle-tracking calculation, Lyapunov exponent analysis

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1. Introduction

Development of particle control technologies by applying an open-ended magnetic field is useful not only for fusion development but also for processing and medical technologies. Regarding nuclear fusion, Momota *et al*. proposed a concept of non-adiabatic trap with the Helmholtz coil inside the solenoid, where the magnetic field by the Helmholtz coil cancels that from the solenoid at the center of confinement region [1]. The coil arrangement of the non-adiabatic trap is schematically drawn in Fig. 1. By creating a region where the magnetic field becomes zero at the center, it can be expected that the adiabatic invariance in a charged particle motion will be broken. Charged particles, in particular ions, will suffer from collisionless pitch-angle scatterings [2–4] and will be able to move all over the accessible region that is specified by the constant of motion such as the Hamiltonian and the canonical angular momentum. By connecting multiple non-adiabatic traps in the axial direction, it becomes possible to extend a net confinement time because of statistical properties shown in random walk processes [1].

Also, a particle control using such a special magnetic field structure is worthy to study. It is expected that ions with smaller axial momentum can be separated from those with larger momentum by the magnetic field in the non-adiabatic trap. Therefore, it can be used as a particle separation apparatus. Adachi *et al*. calculated trajectories of ions in the trap and obtained the trapping ratio [5]. In Ref. [5], beam ions were assumed to be spatially dispersed in a Gaussian distribution when they are drawn out from an ion source. Then, when the magnetic field generated by the solenoid is completely canceled by that from the Helmholtz coil, it was found that the trapping ratio is a function of the standard deviation of the beam dispersion divided by the Larmor radius evaluated by the solenoid magnetic field and the beam ion energy. This gives the conditions under which the beam ion passes through the device. Adachi *et al*. simultaneously examined the particle trapping in the state where the Helmholtz coil current was changed and the magnetic field at the center of the device was not zero. In this case, it has been found that there is an energy band where the trapping rate is relatively low [5] and therefore the ions probably tend to exhibit regular motions. In this study, a Lyapunov exponent analysis [4] is carried out to investigate chaotic properties of ion motions in the situation mentioned above.

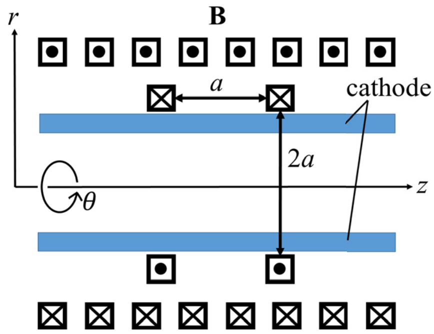

Fig. 1 Schematic drawing of a non-adiabatic trap.

2. Calculation Model

2.1 Device model

A schematic drawing of a device combining a solenoid, a Helmholtz coil, and a cylindrical structure ma-

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Table 1 Calculation model parameters.

Solenoid coil radius	0.2 m
Solenoid coil turns	100 m^{-1}
Solenoid coil current	1.0 kA
Helmholtz coil radius	0.15 m
Helmholtz coil current	15 kA
Calculation region	r: 0.0 to 0.2 m z: -0.25 to 0.25 m

terial such as cathode is shown in Fig. 1. Although the cathode is installed inside the Helmholtz coil, we treat it as a simple structure and no voltage is applied to investigate only influences of the magnetic field. The calculation parameters are shown in Table 1. The current flowing through the Helmholtz coil in Table 1 is not large enough to cancel completely the magnetic field created by the solenoid. The coordinate system is a two-dimensional cylindrical coordinate system. We carry out a particle-tracking calculation for ions and study trapping characteristics of particles in the device shown in Fig. 1. All particles are injected from the end of the apparatus in the axial direction. The ions are classified as trapped and passing particles. The trapped particles are reflected by the mirror field and passing particles reach the opposite end of device without one mirror reflection. When particles hit the cathode, it is assumed that the particles are treated as lost particles. Since we are focusing on particle trapping rate, no discussion on the lost particles will be made here.

The azimuthal component of the vector potential $\mathbf{A}(r, z)$ created by the circular current is

$$A_\theta(r, z) = \sum_i \frac{\mu_0 I_i}{4\pi} \int_0^{2\pi} \frac{\cos\chi \, \mathrm{d}\chi}{\sqrt{(z - z_i)^2 + r^2 + a_i^2 - 2a_i r \cos\chi}}. \tag{1}$$

Here, μ_0 is the vacuum permeability, a_i, z_i and I_i are the radius, the axial position and the circular current of i-th coil, respectively. The integration on χ in Eq. (1) is done numerically. By substituting the obtained vector potential into the following relation: $\mathbf{B} = \mathrm{rot}\mathbf{A}$, the magnetic field can be obtained. In this study, the Helmholtz coil current is set to 15 kA. This Helmholtz coil current does not completely cancel the magnetic field on the midplane and geometric axis generated by the solenoid. Adachi's research [5] showed that the fraction of trapped particles decreases at specific energies of ions in this magnetic field. The purpose of this study is to investigate properties of stochastic motions of ions with energy around the minimum trapping rate. The magnetic field structure when the Helmholtz coil current is 15 kA is shown in Fig. 2. Here, z_M is the half length of the device and it is 0.25 m. It turns out that the x-points where the magnetic field is zero exist slightly inside

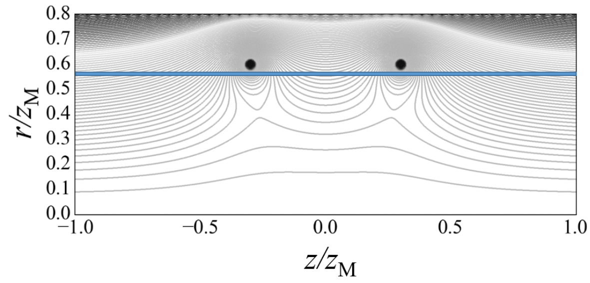

Fig. 2 Magnetic field structure of the non-adiabatic trap. Two black solid circles indicate the position of the Helmholtz coil. Here z_M is 0.25 m.

the Helmholtz coil. Also, it is found that there is the finite axial field along the geometric axis because of incomplete cancellation by the magnetic field from the Helmholtz coil.

2.2 Particle-tracking calculation

Deuterium ion trajectories are traced by integrating the following simultaneous differential equations:

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{v}, \quad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \frac{q}{m}\mathbf{v} \times \mathbf{B}. \tag{2}$$

Here, $\mathbf{x}$ and $\mathbf{v}$ are the position and velocity of ions respectively, and q and m are the charge and mass of ions, respectively. The Runge-Kutta method is employed for numerical integration. The initial position of all ions are z = −0.25 m at the left end of the device, and 10,000 particles are arranged at equal intervals in the radial direction. In order to calculate the trapping rate, it is necessary to give the initial condition of ions with more realistic setting. However, the purpose of our study is to investigate a stochastic behavior of ion motion in this magnetic field structure, and it is preferable that the relation between the initial condition and chaoticity is clear. Therefore, although not a realistic model, the initial positions of the ions are arranged at equal intervals in the radial direction.

3. Lyapunov Exponent Analysis

As a quantity expressing the degree of chaos, the Lyapunov exponent is well-known. In chaotic systems, slightly different two solutions move away each other exponentially with time. Therefore, displacements in the position and velocity can be written generally as follows:

$$|\Delta\chi_i(t)| = |\Delta\chi_i(0)| \exp(\lambda_i t). \tag{3}$$

Here, χ_i is a general description of position or velocity scalar variables, and in the case of three-dimensional space, $i = 1, 2, \cdots, 6$. Note that χ_i is not a subscripted expression of vector variables. The quantity λ_i in Eq. (3) is the Lyapunov exponent for the i-th variable. Using the equation of motion for ions (2), we can write

$$\frac{\mathrm{d}\chi_i}{\mathrm{d}t} = F_i(\chi_j). \tag{4}$$

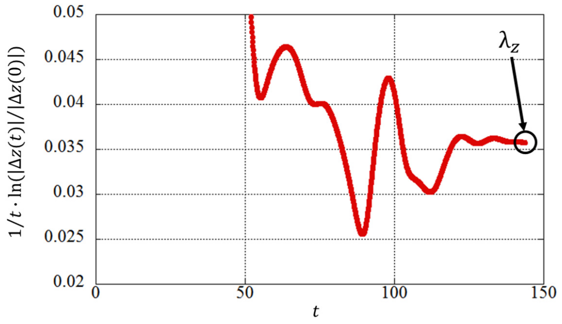

Fig. 3 Example of time evolution of $1/t \cdot \ln(|\Delta z(t)|/|\Delta z(0)|)$.

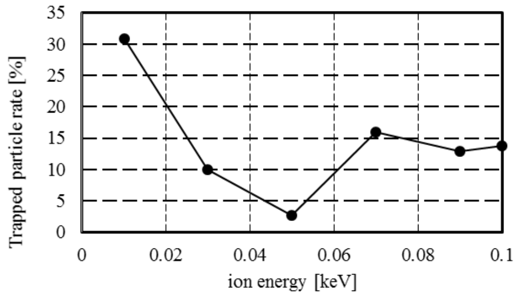

Fig. 4 Relationship between particle trapped rate and ion energy.

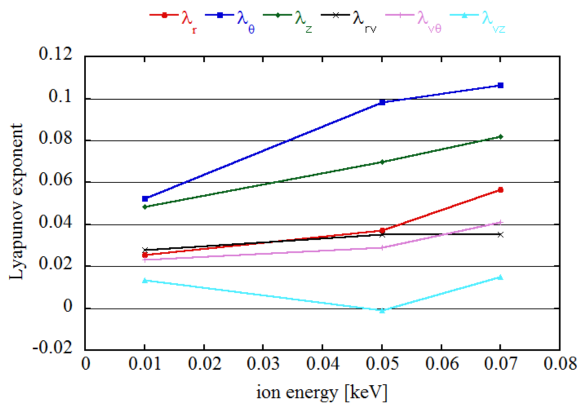

Fig. 5 Relationship between Lyapunov exponents and ion energy.

Here, $F_i(\chi_j)$ is a function given for each component representing the velocity and acceleration. When an infinitesimal displacement $\Delta\chi_i$ is given, then

$$\frac{\mathrm{d}(\chi_i + \Delta\chi_i)}{\mathrm{d}t} = F_i(\chi_j + \Delta\chi_j). \tag{5}$$

Taylor expansion of the right side of Eq. (5) and consideration of Eq. (4) yield

$$\frac{\mathrm{d}\Delta\chi_i}{\mathrm{d}t} = \frac{\partial F_i}{\partial \chi_j}\Delta\chi_j. \tag{6}$$

By numerically integrating the displacement $\Delta\chi_i$ using the Runge-Kutta method as in the trajectory calculation of ions, we also obtain the time evolution of $\Delta\chi_i$. Using Eq. (3), we can estimate the Lyapunov exponent for each variable in the following way:

$$\lambda_i \equiv \lim_{t\to\infty} \frac{1}{t} \ln \frac{|\Delta\chi_i(t)|}{|\Delta\chi_i(0)|}. \tag{7}$$

The value calculated by Eq. (7) converges to a certain value over time, and the steady-state value is the Lyapunov exponent. Usually, however, ions are lost away from the confinement device before the conversion. So in the present study, the final value of the trajectory calculation is adopted as the Lyapunov exponent. Typical time evolution of $1/t \cdot \ln(|\Delta z(t)| / |\Delta z(0)|)$ is shown in Fig. 3, where λ_z for the axial position z is calculated and the time t is normalized by the typical gyration time in the magnetic field generated by the solenoid (i.e., $m/(qB_s) = 0.166\,\mu$s and $B_s = \mu_0 n I_s$ where n and I_s are the turns per unit length and the current of the solenoid, respectively). Although numerical vibration is seen, it is found that λ_z almost converges to 0.036.

4. Results and Discussion

Ions are injected in the axial direction from the end of the device. Here, particles moving through the device are defined as passing particles and particles which are mirror-reflected at least once are defined as trapped particles. Ions colliding with the cathode shown in Figs. 1 or 2 are excluded for our analysis. Figure 4 shows the energy dependence of the fraction of trapped particles. As in the case of Adachi *et al.* [5], it is found that the particle trapped rate becomes a minimum value at 0.05 keV. That is, the 0.05-keV ions are likely to pass through the device without mirror reflection.

Therefore, Lyapunov exponents for six components of position and velocity (λ_r, λ_θ, λ_z, λ_{v_r}, λ_{v_θ}, and λ_{v_z}) are calculated for ions with energy of 0.05 keV (the minimum trapped rate), 0.01 keV and 0.07 keV, and the results are shown in Fig. 5. As a result, it is found that the Lyapunov exponents except λ_{v_z} increase as energy increases. However, λ_{v_z} is the smallest at 0.05 keV. In other words, in the case of 0.05 keV, it is found that the initial value sensitivity to the deviation of the initial axial velocity of the two ions is small.

A typical trajectory of (a) a 0.05-keV passing ion and (b) a trapped ion are drawn and projected onto xy plane as shown in Fig. 6. Here, the axes of coordinate are transformed by $x = r\cos\theta$ and $y = r\sin\theta$ from the cylindrical coordinates shown in Fig. 1. In either case, the trajectory radius increases as the ion passes through the weak magnetic field region. At the same time, the trajectory deviates from a regular circular motion. However, it can be seen that the trapped particles are subject to more changes of the velocity direction, i.e., collisionless pitch angle scatterings [2, 3], which depend on the gyrophase when ions enter and exit the weak magnetic region. For a small trapped rate, ions entering the weak magnetic field region are un-

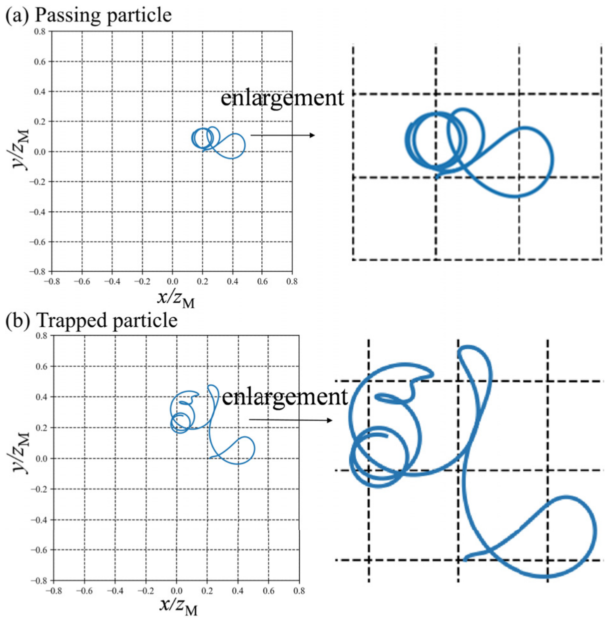

Fig. 6 Typical 0.05-keV ion trajectory projected onto the xy plane of (a) passing particle and (b) trapped particle.

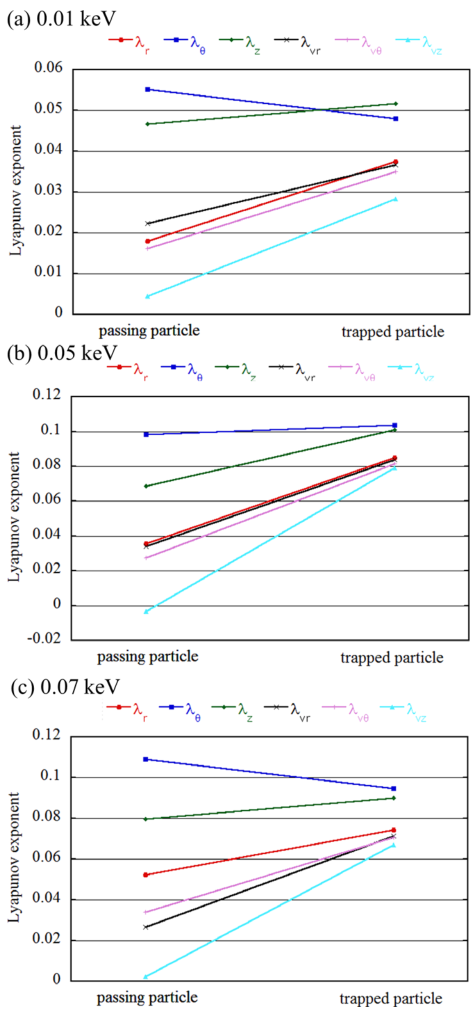

Fig. 7 Lyapunov exponents of passing particles and trapped particles for (a) 0.01 keV, (b) 0.05 keV and (c) 0.07 keV.

likely to cause perturbation of their trajectory and exhibit regular motions. As a result, changes in the velocity component in the axial direction, that is, in the direction along the magnetic field are small, and ions tend to pass to the opposite device end. Therefore, the Lyapunov exponent λ_{v_z} becomes smaller, and it can be understood that the ion motion becomes more regular.

The results of the Lyapunov exponents for (a) 0.01 keV, (b) 0.05 keV, and (c) 0.07 keV ions as passing particles and trapped particles are shown in Fig. 7. As described above, an ion trapping by a non-adiabatic trap is due to a stochastic behavior seen in an ion motion and a consequent large Lyapunov exponent. In other words, it is clear that statistical behavior of ion motion is strongly related to trapping and passing in non-adiabatic traps. Also from Fig. 7, it can be seen that, which is the maximum Lyapunov exponent, tends to have different tendency from other Lyapunov exponents. It is found that λ_θ of trapped particles is larger than that of passing particles only at 0.05 keV at which the trapped rate becomes minimum. On the other hand, λ_θ of trapped particles is smaller at 0.01 keV and 0.07 keV. Here, since the maximum Lyapunov exponent is positive at any energy, the initial value sensitivity is still noticeable in the ion motion. In the case of 0.01 keV or 0.07 keV, the initial value sensitivity of passing particles is pronounced, and ions can be easily trapped due to a slight disorder of orbits, whereas it is found that the tendency is relatively small at 0.05 keV.

5. Conclusion

In this study, ion motion in a non-adiabatic trap, which is a magnetic field structure created by partially canceling a magnetic field from a solenoid by that from a Helmholtz coil, has been investigated. In particular, we have focused on the energy dependence of the trapping rate of beam ions injected into the non-adiabatic trap in the axial direction. The Lyapunov exponent analysis has been carried out on ions with energy (0.05 keV in this study) that minimizes the trapped rate. As a result, it was found that the Lyapunov exponent for the axial velocity is minimized at 0.05 keV. This indicates that 0.05-keV ions have relatively regular motion. Furthermore, it was found that the ions are difficult to undergo pitch angle scattering when passing through the weak magnetic field region. The Lyapunov exponents were compared between passing particles and trapped particles and it was found that the Lyapunov exponent of trapped particles was larger. Therefore it was found that trapping in the device is the result of random motion.

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