Relative Dispersion of Trapped Ion Granulations in Sheared Flows*)

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The life time of trapped ion granulations (trapped ions correlated by resonance) in sheared flows is calculated. The dynamics of trapped ion granulations, in the presence of sheared flows, is formulated in terms of two point correlation function of phase space density fluctuations. The evolution equation is closed by a simplified closure calculation of the triplet term. Based on the closed equation, the life time of the relative dispersion of trapped ion granulations is calculated. The result shows that i.) a relevant time scale enters via a hybrid of decorrelation and shearing, $(\Delta\omega_c v_y'^2)^{1/3}$ and ii.) small scale singularities in the absence of collisional dissipation enters through logarithmic divergence.

Keywords: trapped ion, granulation, sheared flow, relative dispersion, life time

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1. Introduction

Turbulent mixing is an important issue for magnetic fusion community for predictive modeling of anomalous transport (Fig. 1). At the simplest level, the idea of turbulent mixing, or so-called the mixing length theory [1], is applied for estimating saturated level of turbulence. More elaborated theory for turbulent mixing modeling was developed and applied for estimating the anomalous level of transport by ion temperature gradient (ITG) driven turbulence [2], etc. More recently, the effect of sheared flows on the turbulent mixing has been elucidated [3–5]. In the case with sheared flows, while turbulent $E\times B$ eddys mixes background physical quantities, they are sheared apart by flows. The combination of the two processes results in the hybrid mixing rate $(k_0^2 D_\perp v_y'^2)^{1/3}$, where $k_0^2 D_\perp$ is a typical mixing rate due to turbulent $E \times B$ diffusion and v_y' is flow shear. The turbulence and associated transport are suppressed when $(k_0^2 D_\perp v_y'^2)^{1/3} > k_0^2 D_\perp$, in which case turbulent $E\times B$ eddys are sheared apart before causing appreciable transport.

Mixing by plasma turbulence can be also formulated in phase space [6–8]. The mixing process by phase space 'eddys', called granulations (alternatively put, granulations or clumps are macro-particles constituting of resonant particles correlated via resonance), is pioneered by Dupree [6]. In that work, Dupree calculated the rate of turbulent mixing in phase space for unmagnetized, 1d plasmas (Vlasov plasmas). The role of phase space density granulations in relaxation was also discussed in the context of fusion study [9, 10]. In particular, in [10], the mixing rate by trapped ion granulations (resonantly correlated trapped

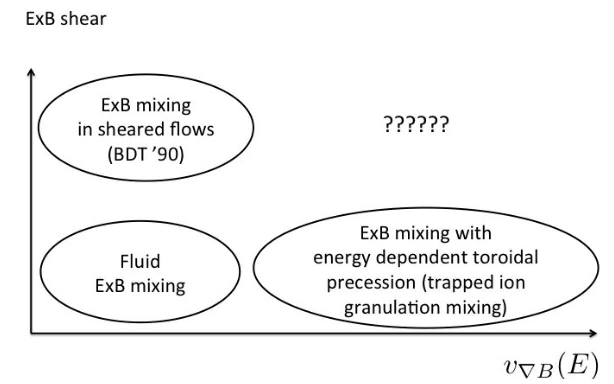

Fig. 1 A schematic view of this study. The idea of turbulent $E \times B$ mixing was applied to predict saturated turbulent amplitude [1] or anomalous transport, say by ITG [2]. The concept has been developed to include shear flow effect in [3, 4]. On the other hand, the $E \times B$ mixing is formulated at the level of phase space by Biglari [10], as shown in the horizontal direction in the cartoon. In this study, we are aiming at developing the $E \times B$ mixing with energy dependent toroidal precession to include shear flow effect.

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ions) are calculated. The calculated mixing rate was applied to formulate nonlinear instability driven by trapped ion granulations and to calculate the level of anomalous transport. However, these were before the role of sheared flows on plasma turbulence were appreciated. Thus these calculations were not formulated in the presence of sheared flows. How sheared flows modifies the mixing by granulations is an important question, since anomalous transport by granulations may be controlled via shear flows.

In this paper, we discuss how the turbulent mixing with granulations is affected by sheared flows. As a typical example of plasma turbulence with granulations, we consider trapped ion resonance driven turbulence. The dynamics of trapped ion granulations is formulated by the evolution of two point phase space density correlation. The life time (mixing rate) of trapped ion granulations are determined by several processes, such as i.) the difference in energy dependent ∇B drift, ii.) the turbulent $E \times B$ scattering, and iii.) shearing by flows. In particular, the second item ($E \times B$ mixing) appears as a triplet term in the evolution equation for phase space density correlation. This term is renormalized to give a turbulent $E\times B$ diffusion term via a simple closure calculation [6, 7]. The closed equation is used to calculate a relevant time scales for the mixing by granulations with sheared flows. The result indicates that the mixing by granulations are modified in the presence of shear flows, in such a way that the relevant mixing rate enters via the hybrid of $E \times B$ mixing rate and flow shear v_y'. The dependence on velocity space variables are retained through logarithmic factor in the mixing time. The result diverges logarithmically as the two points in phase space are approached, which is a typical behavior of turbulence mixing by granulations.

The remaining of the paper is organized as follows. In section 2, we introduce a model used in the paper and derive the evolution equation for trapped ion granulations. A simple calculation based on quasi-linear like theory is presented. In section 3, we calculate a relevant time scale for the mixing by granulations with sheared flows. Section 4 is conclusion.

2. Model Equation and Triplet Closure

Here, we use a model developed in [11] for trapped ion resonance driven turbulence:

$$\partial_t f_i + v_{Di}\bar{E}\partial_y + \boldsymbol{v}_{E\times B}\cdot\nabla f_i = 0. \tag{1}$$

Here $v_{Di}\bar{E}$ is the energy dependent drift velocity and $\bar{E} \equiv E/T_i$. Energy is not scattered in this model due to the bounce average, $\mathrm{d}E/\mathrm{d}t = v_\parallel E_\parallel \to 0$. Electrons are assumed to be dissipative, and thus treated as laminar. Ion density and electron density are connected via the Poisson equation,

$$\frac{\delta n_e}{n_0} = \int \mathrm{d}^3 v f_i + \rho^2 \nabla_\perp^2 \frac{e\phi}{T_e}. \tag{2}$$

Here $\rho^2 = \rho_s^2(1 + 1.6q^2/\sqrt{\epsilon_0})$ includes both classical and neo-classical polarization effects [12]. q is the safety factor and ϵ_0 is the inverse aspect ratio. Arguably, the model here may be the minimal model that captures the drift wave dynamics with the velocity space resonance. Due to the 1d structure of the resonance and the weakly dispersive nature of the long wave length modes as treated in this model, strong resonance such that the Kubo number $K \equiv \tilde{v}\tau_c/\Delta_c \sim \tau_{circ,E\times B}^{-1}(\mathrm{d}\omega/\mathrm{d}k_\theta - \omega/k_\theta)^{-1}\Delta k_\theta^{-1} \gtrsim 1$ is expected to develop even for a broad spectrum $\Delta k_\theta \sim k_\theta$. Thus the formation of phase space structures, such as phase space density granulations, is very likely in this model.

The dynamics of phase space density fluctuations is characterized by the evolution of two point phase space density correlation function:

$$\partial_t \langle \delta f(1)\delta f(2)\rangle + T(1,2) = P(1,2). \tag{3}$$

Here the average is over the center of mass coordinate, $\boldsymbol{x}_+ \equiv (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$. The righthand side represents the source for turbulent fluctuation, $P(1,2) \propto -\langle \tilde{v}_x \delta f\rangle \langle f\rangle'$. Specific form for this term is calculated in [10, 13, 14] and we focus on the lefthand side in the rest of the paper. The lefthand side includes the term that determines the lifetime of the relative dispersion:

$$\begin{aligned} T(1,2) = {} & v_{Di}E_1 \frac{\partial}{\partial y_1}\langle \delta f(1)\delta f(2)\rangle \\ & + v_y' x_1 \frac{\partial}{\partial y_1}\langle \delta f(1)\delta f(2)\rangle \\ & + \nabla_1 \cdot \langle \tilde{\boldsymbol{v}}(1)\delta f(1)\delta f(2)\rangle + (1 \leftrightarrow 2). \end{aligned} \tag{4}$$

Here $(1 \leftrightarrow 2)$ denotes the terms with the arguments 1 and 2 exchanged. Note that this term includes the triplet term due to $E \times B$ mixing. Thus we need a closure calculation. The triplet term can be closed by employing a simple closure modeling based on quasilinear like calculation. In this approach, the triplet term is approximated as:

$$\tilde{\boldsymbol{v}}(1)\delta f(1)\delta f(2) \cong \tilde{\boldsymbol{v}}(1)(\delta f(1)\delta f(2))^c. \tag{5}$$

Here $(\delta f(1)\delta f(2))^c$ is the fluctuating part that is phase coherent to $\tilde{v}_{E\times B}$:

$$\begin{aligned} & (\delta f(1)\delta f(2))^c_{\boldsymbol{k}\omega} \cong -\hat{g}_{\boldsymbol{k}\omega}(1) \\ & \times (\boldsymbol{v}_{E\times B})_{\boldsymbol{k}\omega} \mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{x}_1} \cdot \nabla_1 \langle \delta f(1)\delta f(2)\rangle + (1 \leftrightarrow 2), \end{aligned} \tag{6}$$

and $\hat{g}_{\boldsymbol{k}\omega}(1) = (-i\omega + i\omega_{Di}\bar{E}_1 - k_y v_y' \partial/\partial k_x + 1/\tau_c)^{-1}$ is the propagator. Substituting the phase coherent response, the triplet term then becomes:

$$\begin{aligned} \langle \tilde{\boldsymbol{v}}(1)\delta f(1)\delta f(2)\rangle \cong & -\boldsymbol{D}_{11}\cdot\nabla_1\langle \delta f(1)\delta f(2)\rangle \\ & -\boldsymbol{D}_{12}\cdot\nabla_2\langle \delta f(1)\delta f(2)\rangle. \end{aligned} \tag{7}$$

Here the diffusivity is given by $\boldsymbol{D}_{11} \equiv \sum_{\boldsymbol{k}\omega} \hat{g}_{\boldsymbol{k}\omega} \langle \tilde{\boldsymbol{v}}_{E\times B}\tilde{\boldsymbol{v}}_{E\times B}\rangle_{\boldsymbol{k}\omega}$ and $\boldsymbol{D}_{12} \equiv \sum_{\boldsymbol{k}\omega} \hat{g}_{\boldsymbol{k}\omega} \langle \tilde{\boldsymbol{v}}_{E\times B}\tilde{\boldsymbol{v}}_{E\times B}\rangle_{\boldsymbol{k}\omega} \mathrm{e}^{-i\boldsymbol{k}\cdot\boldsymbol{x}_-}$. Collecting the result and writing the answer in terms of relative

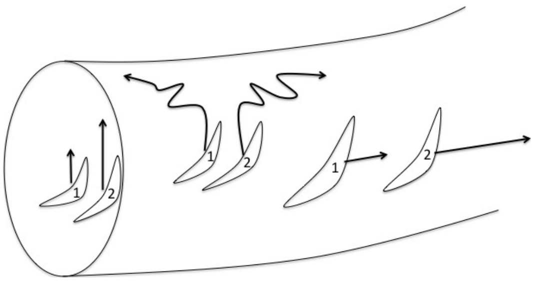

Fig. 2 A schematic view for the processes that determine the life time of trapped ion granulations. Trapped ion granulations are decorrelated due to several processes, including the difference in the precession velocity (right), turbulent $E \times B$ mixing (center), shearing due to flows (left). The effective life time is determined by the combination of these processes.

coordinates($\boldsymbol{x}_- \equiv \boldsymbol{x}_1 - \boldsymbol{x}_2$, $E_- \equiv E_1 - E_2$), we finally have:

$$
\begin{aligned}
T(1,2) =& v_{Di}\bar{E}_- \frac{\partial}{\partial y_-}\langle \delta f(1)\delta f(2)\rangle \\
&+ v'_y x_- \frac{\partial}{\partial y_-}\langle \delta f(1)\delta f(2)\rangle \\
&- \nabla_- \cdot \boldsymbol{D}_- \cdot \nabla_- \langle \delta f(1)\delta f(2)\rangle.
\end{aligned} \tag{8}
$$

Here $\boldsymbol{D}_- \equiv \boldsymbol{D}_{11} + \boldsymbol{D}_{22} - \boldsymbol{D}_{12} - \boldsymbol{D}_{21}$. For small separations $\boldsymbol{k}_0 \cdot \boldsymbol{x}_- < 1$, $\boldsymbol{D}_- \cong \boldsymbol{D}_\perp(\boldsymbol{k}_0 \cdot \boldsymbol{x}_-)^2$ where $\boldsymbol{D}_\perp$ is the typical $E\times B$ mixing diffusivity and $\boldsymbol{k}_0$ is the typical scale for turbulent fluctuations. We are interested in this limit as the condition $\boldsymbol{k}_0 \cdot \boldsymbol{x}_- < 1$ is required for particles to be correlated. Each process that determines $T(1,2)$ is depicted in Fig. 2. The first term is the relative dispersion due to the difference in the precession speeds. The second term is due to shearing. The last term, which is a product of the closure calculation, represents the nonlinear $E \times B$ mixing.

3. Life Time of Trapped Ion Granulations in Sheared Flows

Given the closed equation as derived above, here we extract a life time of phase space density correlations. This may be done by considering the evolution of a probability density function $F(x_-, y_-, E_-)$ for relative separations and by extracting the evolution of moments which are defined as $\langle\langle ... \rangle\rangle = \int \mathrm{d}x_- \mathrm{d}y_- \mathrm{d}E_- (...) F / \int \mathrm{d}x_- \mathrm{d}y_- \mathrm{d}E_- F$. Here F evolves in time as:

$$
\partial_t F + v_{Di}\bar{E}_- \frac{\partial}{\partial y_-}F + v'_y x_- \frac{\partial}{\partial y_-}F - \nabla_- \cdot \boldsymbol{D}_- \cdot \nabla_- F = 0. \tag{9}
$$

To further simplify the analysis, we approximate the diffusion term as

$$
\begin{aligned}
\nabla_- \cdot \boldsymbol{D}_- \cdot \nabla_- \cong& \frac{\partial}{\partial x_-} k_0^2 D_\perp \left(\frac{x_-^2}{k_0^2 \Delta r_c^2} + y_-^2 \right) \frac{\partial}{\partial x_-} \\
&+ \frac{\partial}{\partial y_-} k_0^2 D_\perp \left(\frac{x_-^2}{k_0^2 \Delta r_c^2} + y_-^2 \right) \frac{\partial}{\partial y_-}.
\end{aligned} \tag{10}
$$

Thus we only retain the diagonal terms in the diffusion matrix. Since typical scales in x and y can be anisotropic due to sheared flows, we retained the difference in the scales in x and y, which are given by Δr_c and k_0 respectively. Note that a similar approximation is used in [3]. Using these relations, relevant moments are obtained as:

$$
\partial_t \langle\langle x_-^2 \rangle\rangle = 6\frac{D_\perp}{\Delta r_c^2}\langle\langle x_-^2 \rangle\rangle + 2D_\perp k_0^2 \langle\langle y_-^2 \rangle\rangle, \tag{11}
$$

$$
\begin{aligned}
\partial_t \langle\langle y_-^2 \rangle\rangle =& 2v_{Di}\langle\langle \bar{E}_- y_- \rangle\rangle + 2v'_y \langle\langle x_- y_- \rangle\rangle \\
&+ 2\frac{D_\perp}{\Delta r_c^2}\langle\langle x_-^2 \rangle\rangle + 6D_\perp k_0^2 \langle\langle y_-^2 \rangle\rangle,
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\partial_t \langle\langle x_- y_- \rangle\rangle =& v_{Di}\langle\langle \bar{E}_- x_- \rangle\rangle + v'_y \langle\langle x_-^2 \rangle\rangle \\
&+ 2\frac{D_\perp}{\Delta r_c^2}\langle\langle x_- y_- \rangle\rangle + 2D_\perp k_0^2 \langle\langle x_- y_- \rangle\rangle,
\end{aligned} \tag{13}
$$

$$
\begin{aligned}
\partial_t \langle\langle \bar{E}_- y_- \rangle\rangle =& v_{Di}\langle\langle \bar{E}_-^2 \rangle\rangle + v'_y \langle\langle \bar{E}_- x_- \rangle\rangle \\
&+ 2D_\perp k_0^2 \langle\langle \bar{E}_- y_- \rangle\rangle,
\end{aligned} \tag{14}
$$

$$
\partial_t \langle\langle \bar{E}_- x_- \rangle\rangle = 2\frac{D_\perp}{\Delta r_c^2}\langle\langle \bar{E}_- x_- \rangle\rangle. \tag{15}
$$

The set of equations is solved in strong shear limit, $v'_y \gg D_\perp k_0^2, D_\perp/\Delta r_c^2, k_0 v_{Di}$. In this limit, time asymptotically relevant solution can be obtained as

$$
\begin{aligned}
\langle\langle y_- \rangle\rangle \to \frac{\mathrm{e}^{\sigma t}}{3}\Bigg(& y_-^2 + \frac{\sigma}{\Delta\omega_c}x_-^2 + \sqrt{\frac{2\sigma}{\Delta\omega_c}}x_- y_- \\
&+ \frac{2v_{Di}^2}{\sigma^2}\bar{E}_-^2 + 4\frac{v_{Di}}{\sigma}\sqrt{\frac{\sigma}{2\Delta\omega_c}}\bar{E}_- x_- \\
&+ \frac{2v_{Di}}{\sigma}\bar{E}_- y_- \Bigg),
\end{aligned} \tag{16}
$$

where $\Delta\omega_c \equiv 2k_0^2 D_\perp$ is the turbulent decorrelation rate and $\sigma \equiv (2\Delta\omega_c v'^2_y)^{1/3}$ is the hybrid frequency of the decorrelation rate and flow shear. By noting that the life time of the correlation can be obtained by setting $\langle\langle y(t = \tau_{cl}) \rangle\rangle \sim k_0^{-2}$ (see Fig. 3), the life time of trapped ion granulation can be obtained as

$$
\begin{aligned}
\sigma\tau_{cl} = \ln\Bigg(& \frac{k_0^2 y_-^2}{3} + \frac{k_0^2\sigma}{3\Delta\omega_c}x_-^2 + \frac{k_0^2}{3}\sqrt{\frac{2\sigma}{\Delta\omega_c}}x_- y_- \\
&+ \frac{2v_{Di}^2 k_0^2}{3\sigma^2}\bar{E}_-^2 + \frac{4v_{Di}k_0^2}{3\sigma}\sqrt{\frac{\sigma}{2\Delta\omega_c}}\bar{E}_- x_- \\
&+ \frac{2v_{Di}k_0^2}{3\sigma}\bar{E}_- y_- \Bigg)^{-1}.
\end{aligned} \tag{17}
$$

Eq. 17 is the principal result of the paper. It describes the life time of trapped ion granulations in sheared flows. Physically, it describes how fast the correlations are mixed through the processes as depicted in Fig. 2. As in the case of the turbulent $E \times B$ mixing in sheared flows, the relevant time scale is given by the hybrid of the decorrelation rate and flow shears, $\sigma \sim (\Delta\omega_c v'^2_y)^{1/3}$. It is also characterized by the logarithmic divergence in the limit of small separations, which is the typical behavior of the life time of granulations.

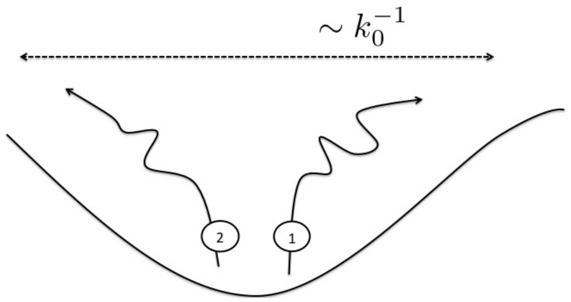

Fig. 3 A cartoon for illustrating the definition of clump life time. When two correlated resonant particles separate to the typical scale of turbulent fluctuation, they are decorrelated. The duration of this process determines the effective life time of the correlation.

4. Conclusion

In this paper, we discussed an extension of formulating the life time of phase space granulations [6] in turbulent plasmas, to include the effect of sheared flows (Fig. 1) [3, 4]. In particular, we focused on trapped ion granulations, a typical example of phase space turbulence which can be important for magnetic fusion community. The life time of trapped ion granulations, formulated first by Biglari [10] for trapped ion resonance driven turbulence without sheared flows, is now extended to include sheared flows. The life time is determined by the combination of several processes, including the difference in the precession velocity, the $E \times B$ turbulent mixing, and shearing by flows (Fig. 2). An analytic expression for the life time is given by Eq. 17. The expression is characterized by; i.) relevant time scale $\sigma = (2\Delta\omega_c v_y'^2)^{1/3}$, which is the hybrid of the decorrelation due to turbulent $E\times B$ mixing and flow shears, and ii.) the logarithmic dependence on the separation of two points in phase space. As a caveat, our calculation does not include the effect of collisions, which do impact the lifetime of phase space structures, even for relatively small, but finite collision frequencies [15].

5. Acknowledgement

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[1] B.B. Kadomtsev, *Plasma Turbulence* (Academic, New York, 1965).
[2] G.S. Lee and P.H. Diamond, Phys. Fluids **29**, 3291 (1986).
[3] H. Biglari, P.H. Diamond and P.W. Terry, Phys. Fluids B **2**, 1 (1990).
[4] T.S. Hahm and K. Burrell, Phys. Plasmas **2**, 1648 (1995).
[5] Ö.D. Gürcan, Phys. Rev. Lett. **109**, 155006 (2012).
[6] T.H. Dupree, Phys. Fluids **15**, 334 (1972).
[7] P.H. Diamond, S.-I. Itoh and K. Itoh, *Modern Plasma Physics Vol.1: Physical Kinetics of Turbulent Plasmas* (Cambridge University Press, Cambridge, 2011).
[8] M. Lesur and P.H. Diamond, Phys. Rev. E **87**, 031101(R) (2013).
[9] P.H. Diamond, P.L. Similon, P.W. Terry, C.W. Horton, S.M. Mahajan, J.D. Meiss, M.N. Rosenbluth, K. Swartz, T. Tajima and R.D. Hazeltine, *Plasma Physics and Controlled Nuclear Fusion Research 1982* (IAEA, Vienna, 1983).
[10] H. Biglari, P.H. Diamond and P.W. Terry, Phys. Fluids **31**, 2644 (1988).
[11] B.B. Kadomtsev and P.O. Pogutse, *Reviews of Plasma Physics*, edited by M. A. Leontovich (Consultants Bureau, New York, 1970) Vol. 5.
[12] Lu Wang and T.S. Hahm, Phys. Plasmas **16**, 062309 (2009).
[13] Y. Kosuga and P.H. Diamond, Plasma Fusion Res. **5**, S2051 (2010).
[14] Y. Kosuga and P.H. Diamond, Phys. Plasmas **18**, 122305 (2011).
[15] M. Lesur, Phys. Plasmas **20**, 055905 (2013).