Relativistic Ponderomotive Force Including Higher Order Nonlocal Effects in High Intensity Laser Fields

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(Received 23 April 2013 / Accepted 2 May 2013)

Based on the noncanonical Lie perturbation theory, we derived a new formula for relativistic ponderomotive force in a transversely localized laser field, which is accessible to the regime where the conventional formula described in terms of the local field gradient can hardly be applied. The formula involves new terms represented by second and third spatial derivatives; therefore, the force depends not only on the local field gradient, but also on the curvature and its variation. A physical explanation for these terms is given.

Keywords: ponderomotive force, nonlocal effect, high intensity laser field, noncanonical Lie perturbation method

DOI: 10.1585/pfr.8.1201094

Ponderomotive force, which corresponds to the pressure of electromagnetic fields, is a central concern in a wide class of nonlinear plasma physics [1]. The force has been derived by the averaging method and formulated as being proportional to the local gradient of the field amplitude [2–4]. An example is that associated with high power lasers in the nonlinear relativistic regime, which are realized by reducing the pulse width and spot size. In such a regime, particles are ejected easily from the interaction region by the ponderomotive force, so that designing laser field patterns is of specifically importance.

A non-Gaussian beam is one of interesting problems. In terms of the interaction between laser and particle, a flat top super-Gaussian profile, which significantly weakens the ponderomotive force near the axis, has an advantage in maintaining a long interaction. The concern is then what determines the particle dynamics in such a case since the force estimated from the conventional formula, which is described only in terms of the local field gradient, tends to be diminished. A residual higher order force associated with nonlocal field profile is predicted, but there exists no formal theory to describe it correctly except direct numerical integration which hardly provides a prospective guideline. In order to find the way out of the difficulty, we herein revisit the ponderomotive force.

The ponderomotive force results from the first order perturbation of the expansion parameter ε, the ratio between particle excursion length and scale length of the field amplitude gradient. In this method, the higher order terms ε^n ($n \geq 2$), which represent the effects of nonlocal particle motion not simply expressed by the local field gradient, are neglected. However, when the local gradient is diminished, the neglected terms can survive and capture the dynamics.

Based on the above idea, we explore the ponderomotive theory including the nonlocal effects up to the order of ε^3. Here, as a method of retaining the Hamiltonian structure up to higher orders, we employ the noncanonical Lie perturbation method based on the variational principle in noncanonical phase space coordinates [5–7]. In a previous study, we applied the method for the first time to derive the ponderomotive force up to the first order of ε and confirmed that the resultant force is consistent with that derived by the averaging method [8]. Based on this approach, the betatron oscillation in a hollow laser field pattern was discussed [9]. In this paper, we extend the analysis to formulate the higher order nonlocal ponderomotive force, in which the nonlocal effect is taken into account by the higher spatial derivatives at the oscillation center.

We consider the linearly polarized laser field given by

$$\boldsymbol{a} = a_x(x)\sin\eta\,\hat{e}_x, \tag{1}$$

where $\boldsymbol{a} \equiv |q|\boldsymbol{A}/mc^2$ is the normalized vector potential, $\eta = \omega t - k_z z$ the phase, q and m the charge and rest mass of the particle, c the speed of light, ω the angular frequency and k_z the wavenumber. In considering the particle motion in the field Eq. (1) in vacuum, we employ the noncanonical phase space coordinate given by

$$z^\mu = (\eta;\ x, y, z, p_x, p_y, p_\eta), \tag{2}$$

where $p_\eta \equiv p_z - \gamma mc$ is a constant of motion in a uniform laser field and γ is the relativistic factor [9]. In this coordinate, the covariant vector γ_μ, by which the variational principle is written as $\delta\int\gamma_\mu \mathrm{d}z^\mu = 0$, is obtained as $\gamma_\mu = \left(-K;\ \boldsymbol{p}_\perp + mc\sigma\boldsymbol{a}_\perp, p_\eta, 0, 0, 0\right)$ where the Hamiltonian K is given by $K = -\left(2k_z p_\eta\right)^{-1}\left(m^2c^2 + \boldsymbol{p}_\perp^2 + p_\eta^2\right)$. Since γ_μ is independent of the coordinates y and z, the corresponding variables p_y and p_η are constants of motion according to the Noether's theorem. The equations of motion

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can be obtained using the variational principle as

$$\frac{\mathrm{d}z^i}{\mathrm{d}z^0} = J^{ij}\left(\frac{\partial \gamma_j}{\partial z^0} - \frac{\partial \gamma_0}{\partial z^j}\right), \tag{3}$$

where J^{ij} is the Poisson tensor defined as the inverse of the Lagrange tensor $\omega_{ij} = \partial_i\gamma_j - \partial_j\gamma_i$. In the uniform laser field, the figure-eight orbit in the x-z plane are derived [9].

Here, to derive the oscillation center equation of motion, we further transform the coordinate z^μ to that including the oscillation center variables, $Z^\mu = (\eta; X, Y, Z, P_x, P_y, p_\eta)$. The relation between the old and new coordinates is defined as $z^i = Z^i + \tilde{z}^{(0)i}$ where $\tilde{z}^{(0)i}$ is the oscillatory component of the figure-eight motion in the uniform laser field. To also express the laser field in terms of the oscillation center variables, we expand $a_x(x)$ in Eq. (1) around $x = X$ as

$$a_x(x) = a_x(X)\left[1 + \varepsilon\frac{\tilde{x}}{L} + \varepsilon^2\frac{\tilde{x}^2}{2!R} + \varepsilon^3\frac{\tilde{x}^3}{3!T} + \cdots\right], \tag{4}$$

where $\tilde{x} \equiv X - x$. Here the expansion parameter ε is defined as $l/L \sim \mathscr{O}(\varepsilon)$ where $l \equiv a_x(X)/k_z\zeta_0$ is the figure-eight excursion length in the x direction, and $L^{-1} \equiv \partial_x \ln a_x(x)|_{x=X} = a_x^{-1}\partial_X a_x$ the scale length of the gradient of the laser field amplitude. Note that ζ_0 is a constant by which the initial value of p_η is defined as $p_{\eta 0} \equiv -mc\zeta_0$. $R^{-1} \equiv a_x^{-1}\partial_X^2 a_x$ and $T^{-1} \equiv a_x^{-1}\partial_X^3 a_x$ are the curvature of the field amplitude and its derivative, respectively, assuming $l^2/R \sim \mathscr{O}\left(\varepsilon^2\right)$ and $l^3/T \sim \mathscr{O}\left(\varepsilon^3\right)$. Note that now all the derivatives are evaluated at the oscillation center X; this differs from Ref. [8] and [9], in which the expansion is taken around a fixed position.

In the perturbation analysis, we consider the Lie transformation $Z^\mu \mapsto Z'^\mu$ which removes higher order oscillations from the 1-form. This requirement is satisfied by choosing $\gamma_i'^{(n)} = 0$ and

$$\gamma_0'^{(n)} = \overline{\left[(\partial_\nu S^{(n-1)})^{(n)} - (g^{(n-1)j}\omega_{j\nu})^{(n)} + C_\nu^{(n)}\right]V^{(0)\nu}}, \tag{5}$$

for $n \geq 1$, where $S^{(n)}$ is the gauge function, $g^{\mu(n)}$ the Lie generator, $C_\mu^{(n)}$ a vector obtained from the lower order calculations, and $V^{(0)\mu}$ the unperturbed flow vector defined by $V^{(0)0} = 1$ and $V^{(0)i}(z) = \mathrm{d}z^{(0)i}/\mathrm{d}z^0$ [5]. The overline indicates the average over one cycle of η. Note that the first and second terms on the right-hand side (RHS) of Eq. (5) originate from the fact that, in the present analysis, quantities including a_x are functions of X owing to the expansion Eq. (4), so that the X derivative of such quantities of $\mathscr{O}(\varepsilon^n)$ may give rise to terms of $\mathscr{O}\left(\varepsilon^{n+1}\right)$, e.g. $\partial_X\Gamma_j^{(n)} = \mathscr{O}(\varepsilon^n) + \mathscr{O}\left(\varepsilon^{n+1}\right)$, due to the ordering $l/L \sim \mathscr{O}(\varepsilon)$.

The 1-form $\Gamma'_\mu\mathrm{d}Z'^\mu$ in which the oscillations are removed by the Lie transformation up to ε^3 yields to

$$\Gamma_0'^{(0)} = -K + \frac{p'_\eta k_z l^2}{4}\alpha^2 - (1+\alpha)\sigma P'_x l\sin\eta, \tag{6}$$

$$\Gamma_0'^{(1)} = 0, \tag{7}$$

$$\Gamma_0'^{(2)} = -\varepsilon^2\frac{l}{16}p'_\eta k_z l\left[A\frac{l^2}{R} + B\frac{l^2}{L^2}\right] + \varepsilon^2\frac{P_x'^2}{p'_\eta k_z}\left[\frac{1}{2}(1+\alpha)\frac{l^2}{R} - \left(\alpha + \frac{1}{4}\right)\frac{l^2}{L^2}\right], \tag{8}$$

$$\Gamma_0'^{(3)} = 0, \tag{9}$$

where $\sigma \equiv q/|q|$, $\alpha\left(p'_\eta\right) \equiv mc\zeta_0/p'_\eta$, $A = \alpha^4 + 4\alpha^3 + 2\alpha^2$ and $B = 7\alpha^4/4 + 8\alpha^3 + 6\alpha^2$. The phase space components are obtained as $\Gamma'_i = \left(P'_x, P'_y, p'_\eta, 0, 0, \Gamma'_6\right)$. Here $\Gamma'_6 = k_z l^2\left(1-\alpha^2\right)\sin(2\eta)/8$ appears due to the gauge transformation $\Gamma_\mu \mapsto \Gamma_\mu + \partial_\mu S^{(0)}$ where $S_0 = p'_\eta k_z l^2\left(1+\alpha^2\right)\sin(2\eta)/8$, which is found to contribute to removing oscillations from the resultant equation of motion, the details will be discussed in a separate paper. Note here that the odd orders of the Hamiltonian, Eqs. (7) and (9), are zero. Then, one can see that the second and fourth order forces do not appear, since the nth order Hamiltonian leads to a force of the $(n+1)$th order as seen from the $(i, j) = (4, 1)$ component in Eq. (3). Here, it is worth considering the general properties of the higher order terms. Since the ponderomotive force is a pressure force associated with electromagnetic fields, it does not depend on the sign of the particle charge σ. Therefore, only the terms proportional to σ^{2n} $(n = 1, 2, \cdots)$ can be retained in the secular 1-form, so that only $\Gamma_0'^{2n}$ has a finite value, which produces the ponderomotive force of the order ε^{2n+1}.

The equations of motion are derived from the 1-form $\Gamma'_\mu\mathrm{d}Z'^\mu$ obtained above. For $i = 2, 5$, and 6, $\mathrm{d}Y'/\mathrm{d}\eta = -P'_y/k_z p'_\eta$, $\mathrm{d}P'_y/\mathrm{d}\eta = 0$ and $\mathrm{d}p'_\eta/\mathrm{d}\eta = 0$ are derived, which lead to $Y' = y_0$, $P'_y = 0$, and $p'_\eta = -mc\zeta_0$, assuming the initial condition $(\boldsymbol{X}'_\perp, \boldsymbol{P}'_\perp) = (\boldsymbol{x}_{\perp 0}, \boldsymbol{0})$ at $\eta = 0$. By using these solutions, in the X' direction, i.e. $i = 1$ and 4, we obtain

$$\frac{\mathrm{d}X'}{\mathrm{d}\eta} = \frac{P'_x}{mc\zeta_0 k_z}\left(1 + \varepsilon^2\frac{3}{2}\frac{l^2}{L^2}\right), \tag{10}$$

$$\frac{\mathrm{d}P'_x}{\mathrm{d}\eta} = -\frac{mca_x}{2}\left[\varepsilon\frac{l}{L} + \frac{\varepsilon^3}{8}\left(\frac{7}{2}\frac{l}{L}\frac{l^2}{R} + \frac{l^3}{T} + \frac{1}{2}\frac{l^3}{L^3}\right)\right]. \tag{11}$$

Here, an additional term proportional to $P_x'^2(l/L)\left(l^2/R\right)$ appears on the RHS of Eq. (11), however, we neglect it since P'_x is of the order ε, so that the term is $\mathscr{O}\left(\varepsilon^5\right)$. Note here that a_x, l, L, R, and T are functions of X'. Equations (10) and (11) determine the transverse secular motion of the oscillation center up to $\mathscr{O}\left(\varepsilon^3\right)$. As seen in Eq. (11), the next order ponderomotive force following the first order is $\mathscr{O}\left(\varepsilon^3\right)$, which consists of the terms proportional to second and third spatial derivatives of the field, and also to the cube of the field gradient. Thus, the force depends not only on the local field gradient, but also on the field curvature and its derivative (spatial variation of curvature) which correspond to higher-order nonlocal structures not simply described by the local gradient. In the Z' direction, the translational motion driven by the light momentum is

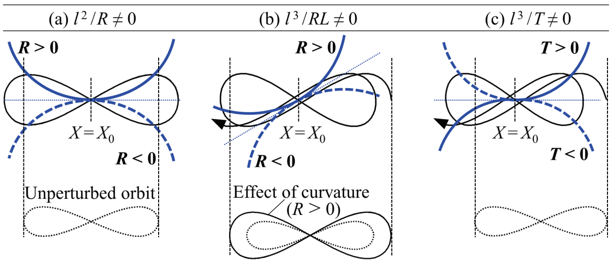

Fig. 1 Effects of the curvature R^{-1} and third derivative T^{-1} of the field amplitude on the particle orbit. The field has (a) a curvature, (b) gradient and curvature, and (c) a curvature transition at $X = X_0$. Black and blue lines represent the particle orbit in the x-z plane and the field pattern, respectively. The unperturbed orbits are shown for comparison below each figure.

also found to be affected by the higher order terms through the $\boldsymbol{v} \times \boldsymbol{B}$ force, which will be shown in a separate paper.

The role of the nonlocal effects can be explained using Fig. 1 which represents three typical laser field patterns, i.e., (a) a symmetrical concave (solid line) or convex (dashed line) structure, i.e. $l/L = l^3/T = 0$ but $l^2/R \neq 0$ at $X = X_0$, (b) an asymmetric concave (solid line) or convex (dashed line) structure where $l/L \neq 0$, $l^2/R \neq 0$ but $l^3/T = 0$, and (c) an asymmetric structure with curvature transition at $X = X_0$ where $l/L = l^2/R = 0$ while $l^3/T \neq 0$. The corresponding particle orbits around $X = X_0$ are shown in Fig. 1 (black solid lines) for $l^2/R > 0$ and $l^3/T > 0$. In case (a), the excursion length increases (decreases) when the curvature is positive (negative) due to the increase (decrease) in the cycle-averaged field amplitude. However, since the change is symmetric for $X = X_0$, the nonlocal effect is cancelled during one cycle of η. Therefore, case (a) does not produce ponderomotive force. This is the reason why the term $l^2/R \sim \mathcal{O}\left(\varepsilon^2\right)$ does not appear independently in Eq. (11). On the other hand, in case (b), the symmetry associated with the curvature l^2/R is broken due to the coupling with the gradient. Consequently, an asymmetry is produced in the orbit, which leads to a ponderomotive force influenced by the curvature. In case (c), the orbit also becomes asymmetric but in a different manner. Namely, although the field gradient is zero at the oscillation center, the nonlocal effect associated with the third derivative yields ponderomotive force. The general parity relation that all the even derivatives, i.e. $\partial^n a_x/\partial X^n$ ($n = 2, 4, 6, \cdots$), do not appear alone in the equation of motion has been confirmed in orders higher than ε^3.

In conclusion, based on the noncanonical Lie perturbation method, we derived a new formula for relativistic ponderomotive force, which depends not only on the local field gradient, but also on the curvature and its variation representing the effect of higher order nonlocal particle motion. The formula is then applicable to the regime where laser fields exhibit characteristic transverse structures such that higher derivatives of the field amplitude regulate the interaction. The formula can provide a theoretical basis not only for understanding the interaction in complicated laser field patterns, but also for designing laser fields by controlling the interaction using nonlocal characteristics of relativistic ponderomotive force. In our next study, we will apply the formula derived here to a specific problem such as a flat-top laser beam.

We are grateful to Drs. K. Imadera and J. Koga for helpful discussions and suggestions. This study was supported by a Grant-in-Aid for JSPS Fellows (No. 24-7688).

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