Analysis of the Relativistic Ponderomotive Force and Higher-Order Particle Motion in a Non-Uniform Laser Field Using the Noncanonical Lie Perturbation Method*)

Natsumi IWATA, Yasuaki KISHIMOTO and Kenji IMADERA

Department of Fundamental Energy Science, Graduate School of Energy Science, Kyoto University, Gokasho, Uji, Kyoto 611-0011, Japan

(Received 6 December 2010 / Accepted 12 April 2011)

Relativistic particle motion in a non-uniform linearly polarized high intensity laser field is analyzed by using the noncanonical Lie perturbation method, which is based on the perturbation theory of the phase space Lagrangian. By introducing the smallness parameter ε as the ratio between the excursion length l and the scale length of the laser field amplitude L, the relativistic ponderomotive force and particle motion are derived up to the second order with respect to ε, which correspond to the nonlocal extension of the conventional ponderomotive force. Specifically, the particle is found to exhibit a betatron-like oscillation with long period characterized by the curvature of the laser field amplitude.

Keywords: noncanonical perturbation method, Lie transformation, high intensity laser field, relativistic ponderomotive force

DOI: 10.1585/pfr.6.2404105

1. Introduction

Recently, the intensity of ultra-short high power lasers has reached the range of 10^{22} W/cm^2. In this regime, electrons irradiated by such lasers exhibit highly relativistic characteristics. These high intensity lasers have opened up various applications such as high-intensity X-ray/neutron sources, compact accelerators, and fast ignition-based laser fusion. In the near future, higher intensities of $10^{23\text{-}26}$ W/cm^2 will be achieved, enabling the exploration of an entirely new regime where new scientific discoveries are anticipated [1, 2]. To realize such high intensities, the reduction of the pulse width and/or the spot size is necessary. In such spatially localized laser fields, the ponderomotive force (light pressure) inevitably exists and plays an important role in the particle dynamics [3, 4].

The relativistic ponderomotive force, which is proportional to the field gradient, has been investigated by applying the averaging method to the equation of motion by introducing the smallness parameter ε (see Sec. 2). However, as the laser field is tightly focused, higher order perturbations such as the spatial curvature become important. Namely, in the non-uniform field, in addition to the force that simply ejects charged particles from the strong field region, particles experience an additional force originating from the curvature of the field. Such a higher-order force may be used to confine particles by carefully controlling the laser field pattern.

In order to investigate particle motion in complicated electromagnetic fields, we have introduced the noncanonical Lie perturbation method based on the phase space Lagrangian [5–7] and derived the ponderomotive force up to the first order of ε [8]. The method is found to be efficient and powerful in determining the particle motion systematically while maintaining the Hamiltonian structure. Motivated by this study, here, we extend the analysis to the higher order particle dynamics, including curvature effects.

We briefly describe the noncanonical Lie perturbation theory in Sec. 2 and apply the method to the orbit analysis in Sec. 3. A summary is given in Sec. 4.

2. Noncanonical Lie Transformation

We consider the motion of a particle with charge q in vacuum irradiated by a linearly polarized high-intensity laser field. The field is assumed to propagate in the z-direction and to be localized in the transverse x- and y-directions. We introduce a smallness parameter, $\varepsilon \sim l/L$, where l and L $(\equiv (\partial_x \log|\boldsymbol{a}|)^{-1} \sim (\partial_y \log|\boldsymbol{a}|)^{-1})$ are the transverse excursion length of particle motion in the uniform laser field and the transverse scale length of the laser field amplitude, respectively. Here, we normalize the vector potential of the laser field $\boldsymbol{A}$ as $\boldsymbol{a} \equiv q\boldsymbol{A}/mc^2$, where m is the rest mass of the particle and c is the speed of light. We express $\boldsymbol{a}$ as

$$\boldsymbol{a}(x, y, \eta) = a_x(x, y, \eta)\hat{\boldsymbol{e}}_x + \varepsilon a_z(\eta)\hat{\boldsymbol{e}}_z, \tag{1}$$

where $a_x(x, y, \eta) \equiv a_{0x}(x, y)\sin\eta$, $a_z(\eta) \equiv a_{0z}\cos\eta$, $\eta \equiv \omega t - k_z z$, and $\hat{\boldsymbol{e}}_x$ and $\hat{\boldsymbol{e}}_z$ are unit vectors in the x- and z-directions, respectively. Note that although a_z is necessary

---

author's e-mail: iwata@center.iae.kyoto-u.ac.jp

*) This article is based on the presentation at the 20th International Toki Conference (ITC20).

to satisfy the Maxwell equations, its contribution is of the order of ε. As discussed in Sec. 3, a_z is found to affect not the secular motion but the oscillatory motion of the particle in the first order of ε. It is also noted that a_z affects second order secular motion but, for simplicity, we neglect it in the analysis presented in Sec. 3.3.

We expand the amplitude of the vector potential around the initial particle position $(x, y, z) = (x_0, y_0, 0)$ as

$$\begin{aligned} a_{0x}(\boldsymbol{x}_\perp) &= a_{0x0} + \varepsilon\tilde{\boldsymbol{x}}_\perp \cdot \partial_{\boldsymbol{x}_\perp} a_{0x}(\boldsymbol{x}_{\perp 0}) \\ &+ \frac{\varepsilon^2}{2}\left[\tilde{x}^2\partial_x^2 a_{0x}(\boldsymbol{x}_{\perp 0}) + \tilde{y}^2\partial_y^2 a_{0x}(\boldsymbol{x}_{\perp 0})\right] \\ &+ \varepsilon^2\tilde{x}\tilde{y}\partial_x\partial_y a_{0x}(\boldsymbol{x}_{\perp 0}) + \mathscr{O}\left(\varepsilon^3\right), \end{aligned} \tag{2}$$

where $a_{0x0} \equiv a_{0x}(\boldsymbol{x}_{\perp 0})$, $\tilde{\boldsymbol{x}}_\perp \equiv \boldsymbol{x}_\perp - \boldsymbol{x}_{\perp 0}$, and $\partial_{\boldsymbol{x}_\perp} a_{0x}(\boldsymbol{x}_{\perp 0}) = \partial a_{0x}(\boldsymbol{x}_\perp)/\partial\boldsymbol{x}_\perp|_{\boldsymbol{x}_\perp = \boldsymbol{x}_{\perp 0}}$.

The perturbation theory of the phase space Lagrangian applied here employs noncanonical variables and the Lie transformation method. We introduce the extended phase space expressed by canonical variables as $z^\mu = (t; \boldsymbol{q}, \boldsymbol{p}_c) = (t; q_x, q_y, q_z, p_{cx}, p_{cy}, p_{cz})$, where the time t is an independent variable. The corresponding covariant vector is given by $\gamma_\mu = (-h; \boldsymbol{p}_c, \boldsymbol{0})$, where h is the relativistic Hamiltonian expressed as

$$h(\boldsymbol{q}, \boldsymbol{p}_c, t) = \sqrt{m^2c^4 + c^2(\boldsymbol{p}_c - mc\boldsymbol{a})^2}. \tag{3}$$

In this paper, we use Latin indices that run from 1 to 6, whereas Greek indices run from 0 to 6. Using this notation, the variational principle written in terms of the Lagrangian L, $\delta\int L\mathrm{d}t = 0$, is expressed as $\delta\int\gamma_\mu\mathrm{d}z^\mu = 0$. We call $\hat{\gamma} \equiv \gamma_\mu\mathrm{d}z^\mu$ a fundamental 1-form. The equation of motion is derived from the variational principle as

$$\frac{\mathrm{d}z^i}{\mathrm{d}z^0} = J^{ij}\left(\frac{\partial\gamma_j}{\partial z^0} - \frac{\partial\gamma_0}{\partial z^j}\right), \tag{4}$$

where J^{ij} is the Poisson tensor defined as the inverse matrix of the Lagrange tensor, $\omega_{ij} \equiv \partial_i\gamma_j - \partial_j\gamma_i$.

Here, we transform the canonical coordinate to that suitable for the analysis. Since the 1-form, $\gamma_\mu\mathrm{d}z^\mu$, is a scalar quantity, the general transformation law from γ_μ to the new covariant vector Γ_μ under an arbitrary coordinate transformation $z^\mu \to Z^\mu$ can be obtained from the relation $\gamma_\mu\mathrm{d}z^\mu = \Gamma_\mu\mathrm{d}Z^\mu$. As a preparatory transformation, we first introduce a noncanonical coordinate

$$z^\mu = (\eta;\ x, y, z, p_x, p_y, p_\eta). \tag{5}$$

Here, $\boldsymbol{p} = \boldsymbol{p}_c - mc\boldsymbol{a}$ is the mechanical momentum, $\boldsymbol{x} = \boldsymbol{q}$, and $p_\eta \equiv p_z - \gamma mc$, where γ is the relativistic factor. Since p_η is the invariant of motion in a uniform field, the orbit analysis becomes easier by taking p_η as one of the coordinate variables. The corresponding covariant vector is then calculated as

$$\gamma_\mu = (-K;\ p_x + mca_x(\boldsymbol{x}_\perp, \eta), p_y, p_\eta + \varepsilon mca_z(\eta), 0, 0, 0), \tag{6}$$

where $K = -\left(2kp_\eta\right)^{-1}\left[m^2c^2 + \boldsymbol{p}_\perp^2 + p_\eta^2\right]$ is the new Hamiltonian. Note that the field $\boldsymbol{a}$ does not explicitly appear in the new Hamiltonian but in the first component of γ_μ, which simplifies the perturbation analysis.

In the Lie perturbation method, we consider a near-identity transformation of the order ε, $z^\mu \to z'^\mu = \exp\left(\varepsilon L^{(1)}\right)z^\mu$, under which the corresponding covariant vector is transformed as $\gamma_\mu \to \gamma'_\mu = \exp\left(-\varepsilon L^{(1)}\right)\gamma_\mu + \partial_\mu S$. Here, S is the gauge function and the operator L is defined to act as $Lf = g^\mu\partial_\mu f$ for a scalar function f and $\left(L\hat{\xi}\right)_\mu = g^\nu\left(\partial_\nu\xi_\mu - \partial_\mu\xi_\nu\right)$ for a 1-form $\hat{\xi}$, where g^μ is the Lie generator of the transformation. By collecting all-order transformations as $\gamma'_\mu = \cdots\exp\left(-\varepsilon^2 L^{(2)}\right)\exp\left(-\varepsilon L^{(1)}\right)\gamma_\mu + \partial_\mu S$, the nth-order transformed covariant vector becomes $\gamma'^{(0)}_\mu = \gamma^{(0)}_\mu$ and $\gamma'^{(n)}_\mu = \partial_\mu S^{(n)} - L^{(n)}\gamma^{(0)}_\mu + C^{(n)}_\mu$ for $n \geq 1$, where $C^{(n)}_\mu$ is a component of the 1-form calculated from results of lower-order calculations. We choose the Lie generator and the gauge function to simplify the new 1-form, $\hat{\gamma}'$. In the analysis presented in Sec. 3, we take a Lie generator that leads to $z'^0 = z^0$ and $\Gamma'^{(n)}_i = 0$ for $n \geq 1$ and use the restriction for S to avoid secularities [5].

3. Orbit Analysis in Laser Fields

In the following, a 1-form of the unperturbed oscillation-center coordinate is derived in Sec. 3.1. Then, on the basis of the oscillation-center 1-form, the particle orbits of the first- and second-order with respect to ε are derived in Sec. 3.2 and Sec. 3.3, respectively.

3.1 One-form for the oscillation-center coordinate

In the coordinate described by Eq. (5), unperturbed particle motion $z^{i(0)}$ is obtained by solving the zeroth-order equations of motion given by Eq. (4) as

$$x^{(0)} = \frac{a_{0x0}}{k_z\zeta_0}(\cos\eta - 1) + x_0, \tag{7}$$

$$y^{(0)} = y_0, \tag{8}$$

$$z^{(0)} = \frac{1}{2k_z\zeta_0^2}\left[\frac{a_{0x0}^2}{2}\left(\eta - \frac{1}{2}\sin 2\eta\right) + \left(1 - \zeta_0^2\right)\eta\right], \tag{9}$$

$$p_x^{(0)} = -mca_{0x0}\sin\eta, \tag{10}$$

$$p_y^{(0)} = 0, \tag{11}$$

$$p_\eta^{(0)} = p_{\eta 0}, \tag{12}$$

under the initial condition of $(\boldsymbol{x}, \boldsymbol{p}, p_\eta) = (\boldsymbol{x}_{\perp 0}, 0, 0, 0, p_{\eta 0})$ and $p_z = p_{z0}$ at $\eta = 0$. Here, we defined ζ_0 as $p_{\eta 0} \equiv -mc\zeta_0$. In this notation, $\zeta_0 = 1$ when the initial momentum of the charged particle is zero, i.e., $p_{z0} = 0$. The particle exhibits the well-known figure-eight orbit in the x-z plane whose oscillation center drifts in the z-direction [9]. The excursion length l is obtained from Eq. (7) as $l = a_{0x0}/k_z\zeta_0$.

Next, to investigate secular motion, we transform the coordinate z^μ (Eq. (5)) to that of the oscillation center of the zeroth-order oscillatory motion, $Z^\mu = (\eta; X, Y, Z, P_x, P_y, p_\eta)$. The old and new coordinates are related as follows:

$$x = X + l\cos\eta, \tag{13}$$
$$y = Y, \tag{14}$$
$$z = Z - \frac{a_{0x0}}{8\zeta_0} l \sin 2\eta, \tag{15}$$
$$p_x = P_x - mca_{0x0}\sin\eta, \tag{16}$$
$$p_y = P_y, \tag{17}$$
$$p_\eta = p_\eta. \tag{18}$$

Then, the new covariant vector is obtained as

$$\begin{aligned}\Gamma_\mu = (-\kappa;\ & P_x + mc(a_x(\boldsymbol{X}_\perp,\eta) - a_{x0}(\eta)),\\ & P_y, p_\eta + \varepsilon mca_z(\eta), 0, 0, 0),\end{aligned} \tag{19}$$

where $a_{x0}(\eta) \equiv a_x(\boldsymbol{X}_{\perp 0}, \eta)$. Here, κ is the new Hamiltonian calculated using the relations Eqs. (13)-(18), which yields

$$\begin{aligned}\kappa = K &+ l\left[P_x + mc\left(a_x(\boldsymbol{X}_\perp,\eta) - a_{x0}(\eta)\right)\right]\sin\eta\\ &+ \frac{a_{0x0}}{4\zeta_0} l\left[p_\eta + \varepsilon mca_z(\eta)\right]\cos 2\eta.\end{aligned} \tag{20}$$

The old Hamiltonian K is now written in terms of new coordinate variables $\boldsymbol{P}_\perp$, p_η and η as $K = -\left[(mc)^2 + (P_x - mca_{x0}(\eta))^2 + P_y^2 + p_\eta^2\right]/2k_z p_\eta$.

For the zeroth order orbit, $Z^{(0)i} = (X^{(0)}, Y^{(0)}, Z^{(0)}, P_x^{(0)}, P_y^{(0)}, p_\eta^{(0)})$, the equations of motion that show the same structure as the canonical Hamilton equations are given by

$$\frac{\mathrm{d}X^{(0)}}{\mathrm{d}\eta} = -\frac{P_x^{(0)}}{k_z p_\eta^{(0)}} + \frac{a_{0x0}}{k_z}\left(\frac{mc}{p_\eta^{(0)}} + \frac{1}{\zeta_0}\right)\sin\eta, \tag{21}$$
$$\frac{\mathrm{d}Y^{(0)}}{\mathrm{d}\eta} = -\frac{P_y^{(0)}}{k_x p_\eta^{(0)}}, \tag{22}$$
$$\frac{\mathrm{d}Z^{(0)}}{\mathrm{d}\eta} = -\left(\frac{K}{p_\eta^{(0)}} + \frac{1}{k_z}\right) + \frac{a_{0x0}}{4\zeta_0} l\cos 2\eta, \tag{23}$$
$$\frac{\mathrm{d}\boldsymbol{P}_\perp^{(0)}}{\mathrm{d}\eta} = \boldsymbol{0}, \tag{24}$$
$$\frac{\mathrm{d}p_\eta^{(0)}}{\mathrm{d}\eta} = 0. \tag{25}$$

Here, the Hamiltonian K retains its functional form but is written in terms of variables $Z^{(0)i}$. Then, the zeroth-order trajectory is obtained as

$$Z^{(0)i} = \left(-l + x_0, y_0, \frac{1}{2k_z\zeta_0^2}\left[\frac{a_{0x0}^2}{2} + \left(1 - \zeta_0^2\right)\right]\eta, 0, 0, p_{\eta 0}\right), \tag{26}$$

which is consistent with that given by Eqs. (7)-(12).

3.2 First-order particle orbit

To analyze first-order motion, we perform a near-identity Lie transformation from the oscillation-center coordinate Z^μ to a new one, $Z'^\mu = (\eta; X', Y', Z', P_x', P_y', p_\eta')$. In the first order, $C_\mu^{(1)}$ is obtained as $C_\mu^{(1)} = \Gamma_\mu^{(1)}$. Then, the new first-order covariant vector is obtained as

$$\Gamma_\mu'^{(1)} = \left(\overline{V^{(0)\mu}\Gamma_\mu^{(1)}}; \boldsymbol{0}, \boldsymbol{0}\right), \tag{27}$$
$$\overline{V^{(0)\mu}\Gamma_\mu^{(1)}} = \frac{mca_{0x0}^2}{2k_z}\frac{mc}{p_\eta'}\left[\frac{X' - x_0 + Y' - y_0}{L}\right]. \tag{28}$$

Here, $L = \left(\partial_x\left[\log a_{0x}(\boldsymbol{X}_{\perp 0})\right]\right)^{-1} = \left(\partial_y\left[\log a_{0x}(\boldsymbol{X}_{\perp 0})\right]\right)^{-1}$ and $V^{(0)\mu}$ is the unperturbed flow vector defined by $V^{(0)0} = 1$, $V^{(0)i}(Z^\mu) = \mathrm{d}Z^{(0)i}/\mathrm{d}Z^0$. Overbars in Eqs. (27) and (28) indicate the average over η-period fast oscillations. Note that all terms including a_z are subtracted in the process of averaging in Eq. (28). Then, the new covariant vector up to the first order is given by $\Gamma_\mu' = \Gamma_\mu^{(0)} + \varepsilon\Gamma_\mu'^{(1)}$. Since $\Gamma_\mu'^{(1)}$ does not contain variables P_x', P_y', and Z', the functional form of the oscillation-center equations of motion for X', Y', and p_η' are the same as those of the zeroth-order, i.e., Eqs. (21), (22), and (25), respectively. The equations of motion for other components are obtained as follows:

$$\frac{\mathrm{d}Z'}{\mathrm{d}\eta} = \left.\frac{\mathrm{d}Z^{(0)}}{\mathrm{d}\eta}\right|_{Z'^\mu} + \frac{m^2c^2}{p_\eta'^2}\frac{a_{0x0}^2}{2k_z}\left[\frac{\tilde{X}' + \tilde{Y}'}{L}\right], \tag{29}$$
$$\frac{\mathrm{d}\boldsymbol{P}_\perp'}{\mathrm{d}\eta} = \frac{mc}{p_\eta'}\frac{mca_{0x0}^2}{2k_z L}\hat{\boldsymbol{e}}_\perp, \tag{30}$$

where $\hat{\boldsymbol{e}}_\perp$ is the unit vector perpendicular to the z-axis, $\tilde{X}' \equiv X' - x_0$, and $\tilde{Y}' \equiv Y' - y_0$. The expression $\mathrm{d}Z^{(0)}/\mathrm{d}\eta|_{Z'^\mu}$ is intended to replace coordinate variables $Z^{(0)i}$ with those of Z'^μ, in Eq. (23). Note that since the first-order backward Lie transformation adds only oscillatory components of motion, we have the relation $\bar{Z}'^i = \bar{Z}^i$. Therefore, the right-hand side of Eq. (30) is the ponderomotive force in the original oscillation-center coordinate. From Eq. (30), we can see that a_z does not appear in the first-order ponderomotive force. This result is physically reasonable, given the following consideration: In the presence of a nonzero value of $a_z \sim \mathcal{O}(\varepsilon)$, the particle oscillates in the z-direction due to the z-component of the electric field, εE_z. This first-order oscillation affects the oscillation in the x-direction through the $\boldsymbol{v} \times \boldsymbol{B}$ force. The ponderomotive force appears as the average force on the oscillating particle owing to the field gradient, which is assumed to be $\mathcal{O}(\varepsilon)$. Thus, terms including a_z will appear as oscillatory terms in the first order and secular terms in the second order. We have confirmed that terms proportional to a_{0z} appear in the first-order oscillatory component in both the x- and z-directions.

3.3 Second-order particle orbit

Next, we analyze second-order particle motion. Here, for simplicity, we consider the case where the field is uniform in the y-direction, i.e., $\partial_y\boldsymbol{a} = \boldsymbol{0}$, although it is straightforward to include the variation. We also neglect the z-component of the vector potential in order to see only the effect of the curvature on particle motion. We transform the coordinate to a new one, $Z''^\mu = (\eta; X'', Y'', Z'', P_x'', P_y'', p_\eta'')$, and calculate the new covariant vector Γ_μ'' using the relation $C_\mu^{(2)} = \Gamma_\mu^{(2)} - L^{(1)}\Gamma_\mu^{(1)} +$

$L^{(1)2}\Gamma_\mu^{(0)}/2$. Then, we have

$$\Gamma_\mu''^{(2)} = \left(\overline{V^{(0)\mu}C_\mu^{(2)}}; \mathbf{0}, \mathbf{0}\right), \tag{31}$$

$$\overline{V^{(0)\mu}C_\mu^{(2)}} = \frac{mca_{0x0}^2}{4k_z}\frac{mc}{p_\eta''}\left[\left(\frac{1}{L^2}+\frac{1}{R}\right)\left(\tilde{X}''^2+\frac{l^2}{4}\right) + \frac{1}{L^2}\frac{3}{16}l\frac{a_{0x0}}{k_z}\frac{mc}{p_\eta''}\right]. \tag{32}$$

Here, $R \equiv \left(\left[\partial_x^2 a_{0x}(x_0)\right]/a_{0x0}\right)^{-1}$ is the scale length of the field curvature. The new covariant vector up to the second order is given by $\Gamma_\mu'' = \Gamma_\mu^{(0)} + \varepsilon\Gamma_\mu''^{(1)} + \varepsilon^2\Gamma_\mu''^{(2)}$. Since the new Hamiltonian, $-\Gamma_0''$, does not contain the variable Z'', the corresponding component p_η'' is found to be constant. Thus, the equations of motion in the x-direction are reduced to

$$\frac{\mathrm{d}X''}{\mathrm{d}\eta} = -\frac{P_x''}{k_z p_{\eta 0}}, \tag{33}$$

$$\frac{\mathrm{d}P_x''}{\mathrm{d}\eta} = -\frac{mca_{0x0}}{2}l\left[\frac{1}{L}+\left(\frac{1}{L^2}+\frac{1}{R}\right)\tilde{X}''\right]. \tag{34}$$

These equations determine the particle motion up to the second order with respect to ε, which varies slowly compared with the period of fast oscillation appearing in the zeroth-order orbit.

In the case $1/L^2 + 1/R \geq 0$, we obtain a slow oscillatory motion given by

$$P_x'' = \alpha\sin\theta\eta + P_{x0}^{(2)}\cos\theta\eta, \tag{35}$$

$$X'' = -\frac{\alpha}{mc}\frac{1}{\theta\zeta_0 k_z}(\cos\theta\eta - 1) + \frac{P_{x0}^{(2)}}{mca_{0x0}}\frac{l}{\theta}\sin\theta\eta + X_0'', \tag{36}$$

where

$$\theta = l\sqrt{\frac{1}{2}\left|\frac{1}{L^2}+\frac{1}{R}\right|}. \tag{37}$$

Here, α is a constant determined by the initial condition as $\alpha = mca_{0x0}\theta\left(1 - l/\left(2L\theta^2\right) - 7l/(8L)\right)$, X_0'' is the initial particle position, and $P_{x0}^{(2)} \sim \mathscr{O}(\varepsilon^2)$ is the initial value of P_x'' calculated by the second-order backward Lie transformation. From Eq. (37), we see that θ, the normalized period to the phase η, increases as the curvature of the field and/or the laser field amplitude increases. Remarkably, the unbounded secular motion originating from the first-order ponderomotive force given in Eq. (30) is changed to the bounded solution, Eqs. (35) and (36), by taking into account the second order curvature terms. This motion corresponds to a betatron oscillation by which the particle is confined to a finite radial region.

In the case $1/L^2 + 1/R < 0$, Eqs. (33) and (34) yield the solution

$$P_x'' = \frac{\alpha + P_{x0}^{(2)}}{2}\mathrm{e}^{\theta\eta} + \frac{-\alpha + P_{x0}^{(2)}}{2}\mathrm{e}^{-\theta\eta}. \tag{38}$$

This solution indicates that the particle is ejected from the region of large laser field amplitude. Taking the expansion of the right-hand side of Eq. (38) by assuming $\theta\eta \sim \mathscr{O}(\varepsilon)$, we obtain

$$P_x'' = \alpha\theta\eta + P_{x0}^{(2)}, \tag{39}$$

$$X'' = \frac{\alpha}{mc}\frac{1}{k_z\zeta_0}\frac{\theta}{2}\eta^2 + \frac{P_{x0}^{(2)}}{mc}\frac{1}{k_z\zeta_0}\eta + X_0''. \tag{40}$$

This solution is consistent with that obtained in Eq. (30) up to the first order although the second order collection is included in Eqs. (39) and (40).

Here, we neglected the z-component of the vector potential, a_z, for simplicity. The inclusion of a_z may cause the modulation of amplitude α and/or period θ, which will be discussed separately.

4. Summary

We derived an equation system describing the relativistic ponderomotive force and the related particle dynamics in a transversely focused linearly polarized laser field up to the second order with respect to ε. In the first order, we obtained the ponderomotive force proportional to the field gradient in the x- and y-directions that is essentially the same as the result in Ref. [8]. In the second order, we found that the particle can exhibit long-period betatron-like oscillatory motion characterized by the curvature of the laser field amplitude. This suggests that the control of the curvature is important in confining the particle and maintaining the laser-particle interaction in transversely localized high-intensity laser fields. The betatron-like oscillation may cause intense radiation, which will be discussed in a future study.

Present results up to the first order and the expansion form, Eqs. (39) and (40), up to the second order are consistent with those obtained by performing the perturbation expansion directly on the equation of motion. However, in the present analysis, nonlocal solutions, Eqs. (35), (36), and (38), are obtained for the first time by the Lie perturbation approach.

In the present study, we consider only the case in vacuum, whereas various additional fields, such as self-induced electromagnetic fields and longitudinal and/or transverse plasma waves, and corresponding forces are incorporated in plasmas. Such fields can be phenomenologically included in the present theoretical framework (e.g., Ref. [6]) although they should be self-consistently determined. These studies remain as future work.

Acknowledgments

This study was supported by a Grant-in-Aid from JSPS (No. 21340171).

[1] G.A. Mourou, T. Tajima and S.V. Bulanov, Rev. Mod. Phys. **78**, 309 (2006).

[2] T. Tajima, Scientific Advisory Committee: Report on the ELI Science (2009).

[3] E.A. Startsev and C.J. McKinstrie, Phys. Rev. E **55**, 7527 (1996).

[4] P. Gibbon, *Short Pulse Laser Interactions with Matter* (Imperial College Press, London, 2005) p.36.

[5] J.R. Cary and R.G. Littlejohn, Ann. Phys. **151**, 1 (1983).

[6] Y. Kishimoto, S. Tokuda and K. Sakamoto, Phys. Plasmas **2**, 1316 (1995).

[7] K. Imadera and Y. Kishimoto, Plasma Fusion Res. **6**, 1201004 (2011).

[8] N. Iwata, K. Imadera and Y. Kishimoto, Plasma Fusion Res. **5**, 028 (2010).

[9] E.S. Sarachik and G.T. Schappert, Phys. Rev. D **1**, 2738 (1970).