Theory of Diamagnetic Signal in Current-Free Stellarators

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The toroidal magnetic flux through the plasma column is calculated analytically for current-free stellarators of arbitrary geometry without assumptions on the plasma shape, aspect ratio, etc. This is done with accuracy sufficient for extracting the contribution due to the finite plasma pressure from this flux. The final result is a formula relating the measured diamagnetic signal with β, the ratio of the plasma pressure to the magnetic pressure. This formula is obtained assuming small β and the relative depth of the magnetic well. These are natural conditions for stellarators, therefore the final result can be recommended for magnetic diagnostics without practical limitations.

Keywords: stellarator, magnetic diagnostics, diamagnetic signal, plasma equilibrium, current-free plasma

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1. Inroduction

Diamagnetic measurements are a traditional diagnostics used for determining the plasma energy content in tokamaks and stellarators [1–5]. Interpretation of the measurements are based on a simple formula originally derived for a circular plasma cylinder [6]

$$2\frac{\delta\Phi}{\Phi_0} = \frac{B_J^2}{B_0^2} - \bar{\beta}, \tag{1}$$

where the measured diamagnetic signal is defined as

$$\delta\Phi = \int_{S_\perp} (\boldsymbol{B} - \boldsymbol{B}_v) d\boldsymbol{S}_\perp, \tag{2}$$

$\boldsymbol{B}$ is the magnetic field, $\boldsymbol{B}_v$ is the vacuum magnetic field, $\Phi_0 = B_0 S_\perp$ with $S_\perp = \pi b^2$ being the transverse cross-section of the plasma column, b is its minor radius, B_0 is the toroidal field, B_J is the poloidal field at the plasma boundary due to the net toroidal current, $\bar{\beta} \equiv 2\overline{p}/B_0^2$ is the ratio of the volume-averaged plasma pressure p to the magnetic field pressure $B_0^2/2$.

It is known that Eq. (1) can be applied to large-aspect-ratio tokamaks with a circular plasma [7]. The same or slightly modified formula has been used for conventional stellarators [8–10], though the validity of Eq. (1) has been proved for straight stellarators only [11,12]. Strictly speaking, for stellarators with current-carrying plasma ($B_J \neq 0$) Eq. (1) should contain an additional term [11, 12]. Here we consider the case when this contribution can be disregarded.

The diamagnetic signal for arbitrary shape and aspect ratio current-free plasma in a toroidal conventional stellarator was calculated analytically in [13]. In analytical theory, when a cylindrical result is generalized for account of the toroidal effects, this is usually done by large-aspect-ratio expansion. It was unexpected that the problem considered in [13] turned out to be solvable without this 'natural' simplification and even without any assumption on the plasma shape. The obtained result [13]

$$\delta\Phi = -\frac{1-\delta_{SH}}{F_b} \int_{V_p} p\, d\tau, \tag{3}$$

was ready for practical use. Similar to (1), it directly relates the measured diamagnetic signal $\delta\Phi$ to the value of interest, the integral on the right hand side over the plasma volume. Here the constant $F_b = 2\pi r B_t$ describes the vacuum toroidal field B_t, r is the radius from the main vertical axis, and δ_{SH} is a small correction related to the Shafranov shift and helical field. Later, numerical calculations [14] for the Large Helical Device (LHD) have demonstrated astonishing accuracy of Eq. (3).

Expression (3) was derived for conventional stellarators with planar circular axis like Heliotron E, CHS, LHD, and similar. There is a growing interest to stellarators of more complex geometry [15, 16]. Examples are Wendelstein 7-X [17], TJ-II [18], HSX [19], Heliotron J [20], CHS-qa [21], and NCSX [22]. Our goal here is a further extension of the analytical theory of diamagnetic measurements on wider class of stellarators. We consider arbitrary 3D toroidal plasma configuration without (or with negligible) net toroidal current and calculate analytically the toroidal magnetic flux through the plasma column for such systems. This is done with accuracy sufficient to extract the contribution into the magnetic flux due to the finite plasma pressure. The final result is the formula relating the measured diamagnetic signal to β. This formula can be used for diagnostic purposes.

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2. General Relations and Assumptions

We start from general equilibrium equations

$$\nabla p = \boldsymbol{j} \times \boldsymbol{B}, \tag{4a}$$

$$\boldsymbol{j} = \mathrm{rot}\boldsymbol{B}, \quad \mathrm{div}\boldsymbol{B} = 0, \tag{4b}$$

where p is the plasma pressure, $\boldsymbol{j}$ is the current density, and $\boldsymbol{B}$ is the magnetic field.

Let us introduce a periodic function f_{H} satisfying the equation

$$\boldsymbol{B}\nabla f_{\mathrm{H}} = \boldsymbol{B}^2 - <\boldsymbol{B}^2>, \tag{5}$$

where the brackets mean the standard flux-surface averaging,

$$<f> \equiv \frac{\mathrm{d}}{\mathrm{d}V}\int_V f \mathrm{d}\tau, \tag{6}$$

and V is the volume inside the toroidal magnetic surface $a(\boldsymbol{r}) = \mathrm{const}$. The vector product of $(\boldsymbol{B} - \nabla f_{\mathrm{H}})$ and the force balance equation (4a) gives

$$(\boldsymbol{B} - \nabla f_{\mathrm{H}}) \times \nabla p - \boldsymbol{j} <\boldsymbol{B}^2> = -\boldsymbol{B}(\boldsymbol{jB} - \boldsymbol{j}\nabla f_{\mathrm{H}}). \tag{7}$$

One can see that the vector on the left hand side is divergence-free. Therefore,

$$\boldsymbol{B}\nabla(\boldsymbol{jB} - \boldsymbol{j}\nabla f_{\mathrm{H}}) = 0, \tag{8}$$

and the previous equation can be rewritten as

$$(\boldsymbol{B} - \nabla f_{\mathrm{H}}) \times \nabla p = \boldsymbol{j} <\boldsymbol{B}^2> - \boldsymbol{B} <\boldsymbol{jB}>. \tag{9}$$

This allows us to express the current density, satisfying both $\mathrm{div}\boldsymbol{j} = 0$ and the equilibrium equation (4a), in the form (for more details see [12, 23])

$$\boldsymbol{j} = \frac{<\boldsymbol{jB}>}{<\boldsymbol{B}^2>}\boldsymbol{B} + [(\boldsymbol{B} - \nabla f_{\mathrm{H}}) \times \nabla A], \tag{10}$$

where A is a surface quantity, $\boldsymbol{B}\nabla A = \boldsymbol{j}\nabla A = 0$, defined as

$$A \equiv \int_b^a \frac{p'}{<\boldsymbol{B}^2>}\mathrm{d}a, \tag{11}$$

the prime means the derivative over a, the label of the magnetic surface (arbitrary shape), and $a = b$ is the plasma boundary. We assume $p(b) = 0$ below.

For current-free plasmas we have

$$\langle \boldsymbol{jB} \rangle = 0. \tag{12}$$

In this case the magnetic field $\boldsymbol{b}$ produced by the current (10) can be explicitly expressed as

$$\boldsymbol{b} = (1 - \mathrm{e}^A)(\boldsymbol{B} - \nabla f_{\mathrm{H}}) + \nabla h. \tag{13}$$

It can be easily shown that this field turns the equation $\mathrm{rot}\boldsymbol{b} = \boldsymbol{j}$ into identity, with $\boldsymbol{j}$ given by (10) and (12). Note that $\boldsymbol{b} = \nabla h$ outside the plasma, and h must be periodic there under condition (12) and assuming that $\boldsymbol{b}$ does not contain a toroidal field from external sources.

Expression (13) is used below for calculating the diamagnetic signal. The periodic function h in (13) must be found from the condition $\mathrm{div}\boldsymbol{b} = 0$. However, it will not be needed here since it will drop out in the calculated integrals.

Note that all relations in this section are valid for general equilibrium described by (4a) and (4b). The only restriction (12), meaning that the net current within each magnetic tube $a(\boldsymbol{r}) = \mathrm{const}$ is zero, is rather natural for stellarators.

Now, in addition to (12), we assume that $\beta \ll 1$, which is also natural for toroidal systems with strong toroidal field. Note that the highest β value ever achieved in stellarators is about 5% [24, 25]. Several years ago this was a level target for the largest stellarators [17, 26]. Just for comparison: the reference operation of ITER is considered with $\beta < 3\%$ [27].

When β is small, one can use a simplified expression for A. From definition (11) it follows that

$$A = \frac{p(a)}{<\boldsymbol{B}^2>} + \int_b^a \frac{p}{<\boldsymbol{B}^2>^2}\frac{\mathrm{d}<\boldsymbol{B}^2>}{\mathrm{d}a}\mathrm{d}a. \tag{14}$$

The second term on the right hand side is small and can be neglected. This can be justified by using the equality

$$\frac{1}{<\boldsymbol{B}^2>}\frac{\mathrm{d}<\boldsymbol{B}^2>}{\mathrm{d}a} = -\frac{p'}{<\boldsymbol{B}^2>} - \frac{\Phi' V''(\Phi)}{V'(\Phi)}, \tag{15}$$

which is easily derived from general integral relations for toroidal current-free equilibrium plasma

$$<\boldsymbol{B}^2> V' = F\Phi', \tag{16}$$

$$p'V' = -F'\Phi', \tag{17}$$

where $F(a)$ is the total poloidal current external to the magnetic surface, and $\Phi(a)$ is the toroidal magnetic flux (for more detail see [12] and [23]). With (15), the last term in (14) can be accurately estimated. The result is

$$A = \frac{p(a)}{<\boldsymbol{B}^2>}\left[1 + O\left(\frac{\beta}{2}, U\right)\right], \tag{18}$$

where U stays for the depth of the magnetic well that comes from the term with $V''(\Phi)$ in (15). In stellarators, the magnetic well is always ‘shallow’ [14–22, 28–30], just several percent.

Since $A \propto \beta$ and β is small,

$$(1 - e^A) <\boldsymbol{B}^2> = - <\boldsymbol{B}^2> A\,[1 + O(A)]. \tag{19}$$

Accordingly,

$$\frac{|\boldsymbol{b}|}{|\boldsymbol{B}|} = O\left(\frac{\beta}{2}\right). \tag{20}$$

These properties are used below to calculate the diamagnetic signal.

3. Toroidal Flux

The total toroidal magnetic flux through the plasma column is

$$\Phi(b) = \frac{1}{2\pi} \int_{V_p} \boldsymbol{B} \nabla \zeta \mathrm{d}\tau. \tag{21}$$

Here the integration is performed over the plasma volume V_p, ζ is the arbitrary variable varying by 2π along the torus. It can be the geometrical toroidal angle or the angle in some flux (or magnetic) coordinates.

The vacuum magnetic field $\boldsymbol{B}_v$ can be expressed as (see, for example, [12])

$$2\pi \boldsymbol{B}_v = F_b \nabla \zeta + \nabla \varphi, \tag{22}$$

where

$$F_b = \oint \boldsymbol{B}_v \mathrm{d}\boldsymbol{l}_t \tag{23}$$

with integration along the closed contour making a complete turn along the torus, and φ is the periodic function. With (22), we can rewrite (21) in the form

$$F_b \Phi(b) = \int_{V_p} \boldsymbol{B}\boldsymbol{B}_v \mathrm{d}\tau. \tag{24}$$

By definition,

$$\boldsymbol{B} = \boldsymbol{B}_v + \boldsymbol{b}. \tag{25}$$

Therefore,

$$\boldsymbol{B}\boldsymbol{B}_v = \boldsymbol{B}_v^2 + \boldsymbol{B}_v \boldsymbol{b} = \boldsymbol{B}_v^2 + \boldsymbol{B}\boldsymbol{b} - \boldsymbol{b}^2, \tag{26}$$

and (24) reduces to

$$F_b \Phi(b) = \int_{V_p} (\boldsymbol{B}_v^2 + \boldsymbol{B}\boldsymbol{b} - \boldsymbol{b}^2) \mathrm{d}\tau. \tag{27}$$

The first term in the integral (27) gives a constant that can be calculated by the known vacuum field. In each particular case it can be done numerically. Here we show several useful relations.

It follows from (22) that

$$2\pi \boldsymbol{B}_v^2 = F_b \boldsymbol{B}_v \nabla \zeta + \boldsymbol{B}_v \nabla \varphi. \tag{28}$$

Integrating this over the plasma volume we obtain

$$\int_{V_p} \boldsymbol{B}_v^2 \mathrm{d}\tau = F_b \langle \Phi_v \rangle_\zeta + \frac{1}{2\pi} \int_{V_p} \boldsymbol{B}_v \nabla \varphi \mathrm{d}\tau, \tag{29}$$

where

$$\Phi_v(\zeta) = \int_{S_\perp} \boldsymbol{B}_v \frac{\nabla \zeta}{|\nabla \zeta|} \mathrm{d}S_\perp, \tag{30}$$

and

$$\langle \Phi_v \rangle_\zeta = \frac{1}{2\pi} \int_0^{2\pi} \Phi_v(\zeta) \mathrm{d}\zeta. \tag{31}$$

The first term in (29) is obtained using the formula

$$\mathrm{d}\tau = \mathrm{d}S_\perp \frac{\mathrm{d}\zeta}{|\nabla \zeta|} \tag{32}$$

for the volume element. Since

$$\int_{V_p} \boldsymbol{B} \nabla \varphi \mathrm{d}\tau = 0, \tag{33}$$

the second term in (29) can be transformed as

$$\int_{V_p} \boldsymbol{B}_v \nabla \varphi \mathrm{d}\tau = -\int_{V_p} \boldsymbol{b} \nabla \varphi \mathrm{d}\tau. \tag{34}$$

It is clear that

$$\left| \frac{\boldsymbol{b} \nabla \varphi}{2\pi \boldsymbol{B}_v^2} \right| = O\left(\frac{\beta}{2} \frac{\tilde{B}}{B} \right), \tag{35}$$

where $\tilde{B}$ stands for the oscillating field described by the function φ in (22). In stellarators this field is a small part of the total field B, so the second term in (29) is a correction of the order much smaller that β. Since we need only terms of the order β with respect to the main term in (27), this correction can be disregarded.

The second term in (27) depends on β through the plasma-generated magnetic field $\boldsymbol{b}$. Using the explicit expression (13) for $\boldsymbol{b}$, we obtain for the integral with $\boldsymbol{B}\boldsymbol{b}$ in (27):

$$\int_{V_p} \boldsymbol{B}\boldsymbol{b} \mathrm{d}\tau = \int_{V_p} (1 - e^A) < \boldsymbol{B}^2 > \mathrm{d}\tau. \tag{36}$$

From this exact equality we obtain with (19) and disregarding the corrections of the order of β and U,

$$\int_{V_p} \boldsymbol{B}\boldsymbol{b} \mathrm{d}\tau = -\int_{V_p} p \mathrm{d}\tau. \tag{37}$$

The last term in (27) is a correction of the next order in β, see (20). Combining (27), (29) and (37), we obtain finally the desired result

$$\Phi(b) - \langle \Phi_v \rangle_\zeta = -\frac{1}{F_b} \int_{V_p} p \mathrm{d}\tau. \tag{38}$$

For conventional stellarators this gives us Eq. (3). Recall that (38) is derived here for more complex geometry.

4. Conclusion

Equation (38) shows a direct way of measuring the stored plasma energy. Being similar to relation (1), this can be used for wider family of stellarators. The result is obtained by the expansion in β and contains just the leading term. The higher-order corrections, if needed, can be calculated by combining the methods described here and in [13]. At small β, however, this can be of academic interest only. More important may be incorporation of the effects of plasma anisotropy which can change the relations (3) and (38), as implied by the results [4] for LHD. Extension of our analysis to this case requires replacement of (4a) by the proper force-balance equation. At present there is no yet generally accepted model for description of anisotropic plasma equilibrium in stellarators [4].

The temporal variation of the toroidal flux through the plasma can be measured using the standard technique [3–5, 8–10] or the double loop method [31].

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