# Numerical Matching Scheme for Linear Magnetohydrodynamic Stability Analysis

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A new matching scheme for linear magnetohydrodynamic (MHD) stability analysis is proposed in a form offering tractable numerical implementation. This scheme divides the plasma region into outer regions and inner layers, as in the conventional matching method. However, the outer regions do not contain any rational surface at their terminal points; an inner layer contains a rational surface as an interior point. The Newcomb equation is therefore regular in the outer regions. The MHD equation employed in the layers is solved as an evolution equation in time, and the full implicit scheme is used to yield an inhomogeneous differential equation for space coordinates. The matching conditions are derived from the condition that the radial component of the solution in the layer is smoothly connected to those in the outer regions at the terminal points. The proposed scheme is applied to the linear ideal MHD equation in a cylindrical configuration, and is proved to be effective from the viewpoint of a numerical scheme.



Keywords: MHD equation, Newcomb equation, rational surface, inner layer, outer region, matching condition, implicit scheme, tokamak

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The method of asymptotic matching is well known in magnetohydrodynamic (MHD) stability analysis for a high-temperature, magnetically confined plasma, such as a tokamak [1, 2]. By assuming that an MHD mode is close to the marginal stability, the method divides the plasma region into outer regions and inner layers. An outer region is far from any rational surfaces for the MHD mode considered; the inertial effects there, as well as other dissipative effects, such as resistivity, can be neglected. The mode is then described by the Newcomb equation, an inertia-free linear ideal MHD equation [3]. On the other hand, the inner layer is a thin layer around a rational surface, where all effects should be retained in the MHD equation employed in the analysis. The width of a layer is determined by the plasma inertia (growth rate of an unstable ideal MHD mode) or the plasma resistivity; it approaches zero as the growth rate or the resistivity reduces to zero. By stretching the time and the radial coordinate in the layer, the MHD equation can be reduced to the so-called inner layer equation [4, 5]. Asymptotic matching of the solution of the inner layer equation with that of the Newcomb equation determines the stability of the ideal or resistive MHD mode [6, 7]. Several numerical methods for solving the inner layer equation have been reported [8, 9]. Several authors have developed computer codes for solving the Newcomb equation for one-dimensional [6, 10] and two-dimensional configurations [11–13].

However, a computer code based on asymptotic matching has not been successfully developed for routine use in MHD stability analysis of tokamaks, although the method itself was published in 1963 [1]. The Newcomb equation is singular at each rational surface where two independent solutions exist; one (dominant solution) is not square integrable and is strongly singular; the other (subdominant solution) is square integrable. The ratio of the subdominant solution to the dominant solution is called matching data, and plays a crucial role in determining the behavior of the MHD mode [6, 8]. The numerical experiments have revealed that the matching data are sensitive to the *local* accuracy of the MHD equilibrium and also the *local* mesh structure at rational surfaces. Therefore, determining the stability of MHD modes based on the matching data thus obtained is difficult, or even intractable. These seem to be the main reasons for the lack of such computer codes. The matching approach is nevertheless a potential numerical method for MHD stability analysis, since it promises a fast executing program that can routinely provide stability analysis.

In the present paper, we propose a new matching scheme that is tractable from the numerical viewpoint. We illustrate the underlying idea with the simplest MHD equation — the linear ideal MHD equation in a cylindrical configuration $r \in (0, a)$, where $r$ denotes the radial coordinate and $a$ the plasma radius. The unknown variable is the displacement vector $\boldsymbol{\zeta}(r)$ [14, 15]. We divide the interval $(0, a)$ into three sub-intervals: $(0, r_L)$, $(r_L, r_R)$, and $(r_R, a)$; $(r_L, r_R)$ is a thin interval that contains the rational surface. The other sub-intervals are regarded as the outer regions. We solve the Newcomb equation in $(0, r_L)$ and $(r_R, a)$, which

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is the differential equation on $y(r)$, the radial component of $\boldsymbol{\zeta}(r)$. In the interval $(r_{\rm L}, r_{\rm R})$, which is the inner layer in the present approach, the evolution equation is solved with no approximation. We apply the full implicit scheme to the evolution equation, and obtain an inhomogeneous linear differential equation with respect to the space coordinate $r$. We solve it using a finite element method [15]. And we derive the matching conditions by imposing on the solutions the condition that $y(r)$ in the layer should be smoothly connected to $y(r)$ in the outer regions at $r = r_{\rm L}$ and $r = r_{\rm R}$.

To develop the matching scheme, we employ the well-known linear ideal MHD equation

$$\rho\partial_t^2\boldsymbol{\zeta}(r,t) = \mathbf{F}[\boldsymbol{\zeta}(r,t)], \tag{1}$$

in the cylindrical coordinate system $(r,\theta,z)$ with $0 \le r \le a$; the infinitesimal displacement of plasma is assumed to be $\boldsymbol{\zeta}(r,t)\exp(im\theta - ikz)$, the density of plasma is $\rho$, and the force operator is $\mathbf{F}$ [14]. For simplicity, we assume the fixed boundary condition $y(a,t) = 0$, where $y(r,t)$ is the radial component of $\boldsymbol{\zeta}(r,t)$. We solve Eq. (1) around layer $(r_{\rm L}, r_{\rm R})$, where the inertial term cannot be neglected, using the full implicit scheme

$$\rho\boldsymbol{\zeta}^{n+1} - (\Delta t)^2\mathbf{F}[\boldsymbol{\zeta}^{n+1}] = \mathbf{S}_{\rm in}[\boldsymbol{\zeta}], \tag{2}$$

$$\mathbf{S}_{\rm in}[\boldsymbol{\zeta}] = \rho(2\boldsymbol{\zeta}^n - \boldsymbol{\zeta}^{n-1}), \tag{3}$$

where $\boldsymbol{\zeta}^n(r) = \boldsymbol{\zeta}(r, n\Delta t)$ and $\Delta t$ is the time step in the scheme. Equation (2) reduces to an inhomogeneous linear differential equation of second order in $y^{n+1}$, the radial component of $\boldsymbol{\zeta}^{n+1}$, which should be solved with appropriate boundary conditions. The two other components of $\boldsymbol{\zeta}^{n+1}$ can be expressed by the linear relations in $y^{n+1}$ and $\mathrm{d}y^{n+1}/\mathrm{d}r$; this is the characteristic feature of the ideal MHD equation, and is fully exploited in the present formulation. In the outer regions, by neglecting the inertia term we have

$$\mathbf{F}[\boldsymbol{\zeta}(r,t)] = 0. \tag{4}$$

Equation (4) reduces to the Newcomb equation, which is a linear differential equation of the second order for $y(r)$

$$\mathcal{L}[r, \mathrm{d}/\mathrm{d}r]y(r) = 0. \tag{5}$$

First, we solve Eq. (2) in the interval $(r_{\rm L}, r_{\rm R})$. Let $\mathbf{G}_{\rm in,L}(r)$ be the solution of the homogeneous equation

$$\rho\boldsymbol{\zeta} - (\Delta t)^2\mathbf{F}[\boldsymbol{\zeta}] = 0, \tag{6}$$

under the boundary conditions

$$G_{\rm in,L}(r_{\rm L}) = 1, \quad G_{\rm in,L}(r_{\rm R}) = 0, \tag{7}$$

where $G_{\rm in,L}(r)$ is the radial component of $\mathbf{G}_{\rm in,L}(r)$. Similarly, let $\mathbf{G}_{\rm in,R}(r)$ be the solution of Eq. (6), whose radial component satisfies the boundary conditions

$$G_{\rm in,R}(r_{\rm L}) = 0, \quad G_{\rm in,R}(r_{\rm R}) = 1. \tag{8}$$

Next let $\mathbf{H}_{\rm in}^{n+1}(r)$ be the solution of Eq. (2) under the boundary conditions

$$H_{\rm in}^{n+1}(r_{\rm L}) = 0, \quad H_{\rm in}^{n+1}(r_{\rm R}) = 0, \tag{9}$$

where $H_{\rm in}^{n+1}(r)$ is the radial component of $\mathbf{H}_{\rm in}^{n+1}$. Using $\mathbf{G}_{{\rm in},p}(r)$ $(p = {\rm L}, {\rm R})$ and $\mathbf{H}_{\rm in}^{n+1}(r)$, the solution of Eq. (2), whose radial component has the values $\xi_{\rm L}^{n+1}$ at $r_{\rm L}$ and $\xi_{\rm R}^{n+1}$ at $r_{\rm R}$, is given by

$$\begin{aligned}\boldsymbol{\zeta}_{\rm in}^{n+1}(r) = \xi_{\rm L}^{n+1}\mathbf{G}_{\rm in,L}(r) + \xi_{\rm R}^{n+1}\mathbf{G}_{\rm in,R}(r) \\ + \mathbf{H}_{\rm in}^{n+1}(r).\end{aligned} \tag{10}$$

The radial component of $\boldsymbol{\zeta}_{\rm in}^{n+1}(r)$ is

$$\begin{aligned}y_{\rm in}^{n+1}(r) = \xi_{\rm L}^{n+1}G_{\rm in,L}(r) + \xi_{\rm R}^{n+1}G_{\rm in,R}(r) \\ + H_{\rm in}^{n+1}(r).\end{aligned} \tag{11}$$

Let us now solve the Newcomb equation — Eq. (5) in $(r_{\rm R}, a)$. The solution can be expressed as

$$y_{\rm R}(r,t) = \xi_{\rm R}(t)\,G_{\rm out,R}(r), \tag{12}$$

where $\xi_{\rm R}(t)$ is the *undetermined* constant, and $G_{\rm out,R}(r)$ is the solution of Eq. (5) under the boundary conditions

$$G_{\rm out,R}(r_{\rm R}) = 1, \quad G_{\rm out,R}(a) = 0. \tag{13}$$

Note that $G_{\rm out,R}(r)$ is regular and square integrable in the interval $[r_{\rm R}, a] = \{r : r_{\rm R} \le r \le a\}$, since the rational surface is not contained in $[r_{\rm R}, a]$. According to the ideal MHD equation, the other two components of $\boldsymbol{\zeta}$ can be expressed by the linear relations in $y$ and $\mathrm{d}y/\mathrm{d}r$. Consequently, the solution of Eq. (4) is expressed as

$$\boldsymbol{\zeta}_{\rm R}(r,t) = \xi_{\rm R}(t)\mathbf{G}_{\rm out,R}(r), \tag{14}$$

where the vector function $\mathbf{G}_{\rm out,R}(r)$ is constructed from $G_{\rm out,R}(r)$. The outer solution in $(0, r_{\rm L})$ is similarly constructed. Let us write the solution of the Newcomb equation in $(0, r_{\rm L})$ as

$$y_{\rm L}(r,t) = \xi_{\rm L}(t)\,G_{\rm out,L}(r), \tag{15}$$

using the undetermined constant $\xi_{\rm L}(t)$ and the solution $G_{\rm out,L}(r)$ of the Newcomb equation under the boundary condition

$$G_{\rm out,L}(r_{\rm L}) = 1, \tag{16}$$

and the regularity condition at $r = 0$. Then, the solution of Eq. (4) in $(0, r_{\rm L})$ is given by

$$\boldsymbol{\zeta}_{\rm L}(r,t) = \xi_{\rm L}(t)\,\mathbf{G}_{\rm out,L}(r), \tag{17}$$

where the vector function $\mathbf{G}_{\rm out,L}(r)$ is constructed from $G_{\rm out,L}(r)$.

From this procedure for constructing solutions, we already have

$$y_{\rm in}^{n+1}(r_{\rm L}) = y_{\rm L}^{n+1}(r_{\rm L}), \quad y_{\rm in}^{n+1}(r_{\rm R}) = y_{\rm R}^{n+1}(r_{\rm R}), \tag{18}$$

where $y_p^{n+1}(r) = y_p(r,(n+1)\Delta t)$ $(p = \mathrm{R,L})$. We further impose the condition, which is the matching condition in the present scheme, that

$$\left.\frac{\mathrm{d}y_{\mathrm{in}}^{n+1}}{\mathrm{d}r}\right|_{r_{\mathrm{L}}} = \left.\frac{\mathrm{d}y_{\mathrm{L}}^{n+1}}{\mathrm{d}r}\right|_{r_{\mathrm{L}}}, \tag{19}$$

$$\left.\frac{\mathrm{d}y_{\mathrm{in}}^{n+1}}{\mathrm{d}r}\right|_{r_{\mathrm{R}}} = \left.\frac{\mathrm{d}y_{\mathrm{R}}^{n+1}}{\mathrm{d}r}\right|_{r_{\mathrm{R}}}. \tag{20}$$

This condition yields a linear equation on $\xi_{\mathrm{L}}^{n+1}$ and $\xi_{\mathrm{R}}^{n+1}$ (the symbol $'$ stands for $\mathrm{d}/\mathrm{d}r$):

$$\mathbf{A}\begin{bmatrix} \xi_{\mathrm{L}}^{n+1} \\ \xi_{\mathrm{R}}^{n+1} \end{bmatrix} = \begin{bmatrix} (H_{\mathrm{in}}^{n+1})'(r_{\mathrm{L}}) \\ (H_{\mathrm{in}}^{n+1})'(r_{\mathrm{R}}) \end{bmatrix}, \tag{21}$$

where

$$\mathbf{A} = \begin{bmatrix} G'_{\mathrm{out,L}}(r_{\mathrm{L}}) - G'_{\mathrm{in,L}}(r_{\mathrm{L}}) & -G'_{\mathrm{in,R}}(r_{\mathrm{L}}) \\ -G'_{\mathrm{in,L}}(r_{\mathrm{R}}) & G'_{\mathrm{out,R}}(r_{\mathrm{R}}) - G'_{\mathrm{in,R}}(r_{\mathrm{R}}) \end{bmatrix}. \tag{22}$$

The linear equation, and therefore the matching problem, is easily solved, as long as $\det\mathbf{A} \neq 0$; the condition $\det\mathbf{A} = 0$ relates to the eigenvalue problem

$$\rho\lambda\boldsymbol{\zeta}(r,t) = \mathbf{F}[\boldsymbol{\zeta}(r,t)], \tag{23}$$

and this interesting aspect will be discussed in a forthcoming paper. Notice that it is $\mathbf{H}_{\mathrm{in}}(r)$ in the thin layer that should be updated at every time step. It is not necessary to update $\mathbf{G}_{\mathrm{out},p}(r)$ and $\mathbf{G}_{\mathrm{in},p}(r)$ $(p = \mathrm{R,L})$; the matrix $\mathbf{A}$ is constant in time, and new $\xi_{\mathrm{L}}^n$, $\xi_{\mathrm{R}}^n$ are obtained from Eq. (21) for the updated $H_{\mathrm{in}}(r)$. This saves substantial computational costs, in terms of both memory size and CPU time.

The finite element method uses linear elements for Eqs. (2) and (4). The following examples are computed for a uniform mesh with a fine mesh size of $4\times10^{-6}$ to capture the structure of a weakly unstable MHD mode, whereas the time step is large; $\Delta t = 0.1\omega_{\mathrm{pa}}$.

Figure 1 illustrates the present scheme applied to the $m = 1$ internal kink mode for $2\pi R_0 = 60$ $(k = 2\pi/R_0)$. The safety factor $q$ is shown by the dotted-dashed line. The $q = 1$ surface locates at $r = 0.4$. The matching points are $r_{\mathrm{L}} = 0.35$ and $r_{\mathrm{R}} = 0.45$ $(\Delta r := r_{\mathrm{R}} - r_{\mathrm{L}} = 0.1)$. The "∘" symbols denote the internal kink mode, obtained by globally solving the linear ideal MHD equation Eq. (1) in its full range (0, 1). The inner layer solution is shown by the solid line, and the outer solutions are shown by dotted lines. The solutions constructed by the matching procedure are indistinguishable from those from the global solution. The enlargements around the matching points show that the inner layer solutions connect smoothly to each outer solution.

Figure 2 shows the time evolution of $\xi_{\mathrm{L}}(t)$, indicated by the broken line, and $\xi_{\mathrm{R}}(t)$, indicated by the dotted line, whereas the solid line indicates the time evolution of the norm of the global solution. $\xi_{\mathrm{L}}(t)$ and $\xi_{\mathrm{R}}(t)$ grow with the growth rate $\gamma = 2.7\times10^{-2}\omega_{\mathrm{pa}}$, which is close to the growth rate estimated from the norm of the global solution, $\gamma = 2.6\times10^{-2}\omega_{\mathrm{pa}}$ ($\omega_{\mathrm{pa}}$: Alfven frequency at the plasma surface).

It is important to study the effects of $\Delta r$, the width of

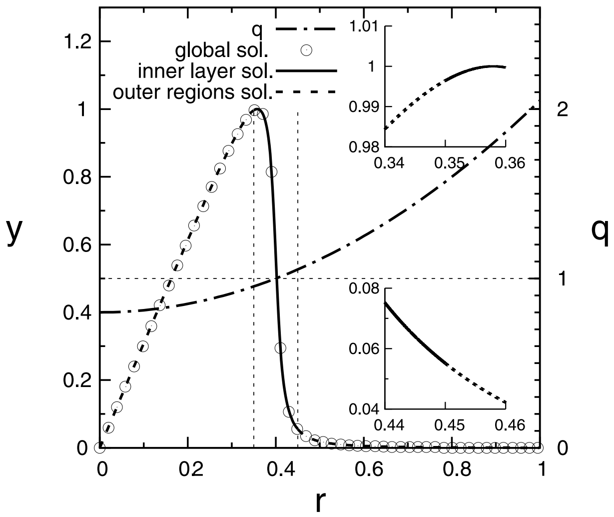


Fig. 1 The radial displacement of the $m = 1$ internal kink mode for $2\pi R_0 = 60$ $(k = 2\pi/R_0)$. The safety factor $q$ is shown by a dotted-dashed line; the $q = 1$ surface is located at $r = 0.4$. The outer and inner layer solutions using the new scheme are indistinguishable from the solution in its full range (0, 1); $r_{\mathrm{L}} = 0.35$, $r_{\mathrm{R}} = 0.45$. Enlargements around the matching points show that each inner layer solution connects smoothly to each outer solution.

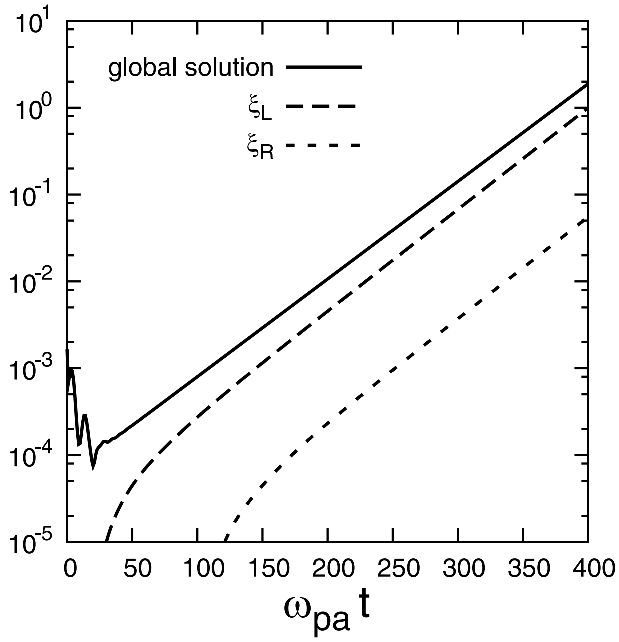


Fig. 2 The evolutions of $\xi_{\mathrm{R}}$ (dotted line), $\xi_{\mathrm{L}}$ (broken line), and the norm of the global solution (solid line). The estimated growth rates of $\xi_{\mathrm{R}}, \xi_{\mathrm{L}}$ are $\gamma = 2.7\times10^{-2}\omega_{\mathrm{pa}}$, and are close to that from the global solution, $\gamma = 2.6\times10^{-2}\omega_{\mathrm{pa}}$ ($\omega_{\mathrm{pa}}$ denotes the Alfven frequency at the plasma surface).

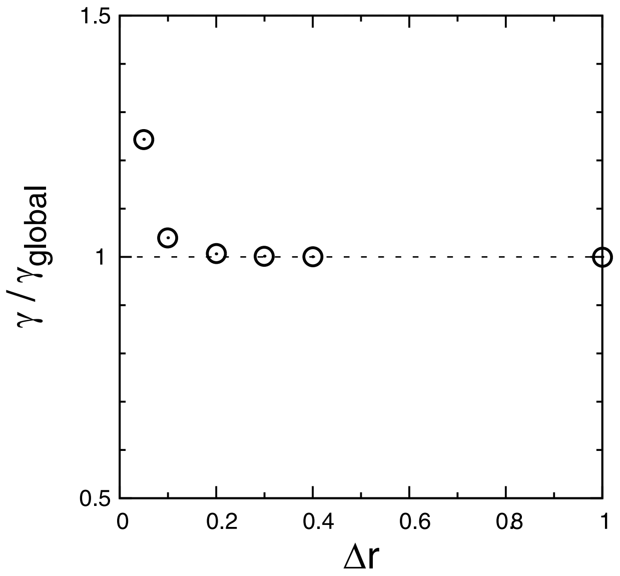


Fig. 3 $\Delta r$ dependence of the growth rate $\gamma$ of the solution by the matching scheme; the growth rate is normalized to $\gamma_{\mathrm{global}}$, the growth rate by the global solution. $\gamma$ agrees well with $\gamma_{\mathrm{global}}$ for $\Delta r \geq 0.2$. When $\Delta r \leq 0.1$, we have $\gamma > \gamma_{\mathrm{global}}$; it is understood that the kinetic energy is underestimated for the solution by the matching scheme with a too thin layer.

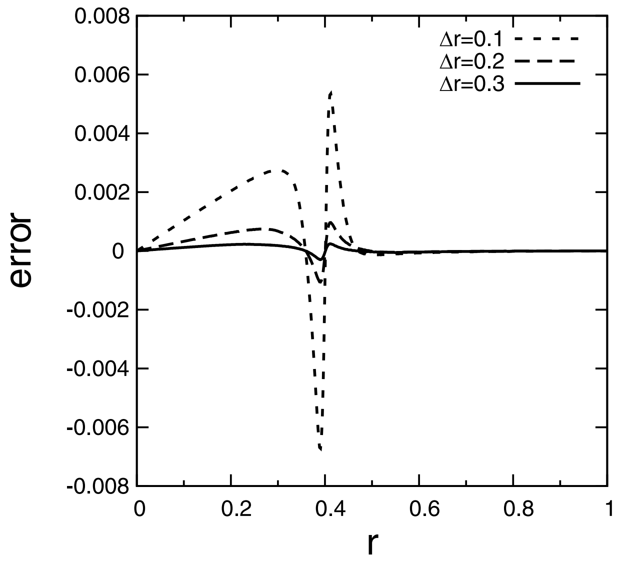


Fig. 4 The matching scheme error of the solution $y(r)$ from the global solution; the solid, broken, and dotted lines are for $\Delta r = 0.3$, 0.2, and 0.1, respectively. The agreement between $y(r)$ and the global solution is already good for $\Delta r = 0.1$. The error profile is unsymmetrical with respect to $r = 0.4$, the position of $q(r) = 1$. The magnetic shear is stronger on the right side of $q > 1$ than the left side of $q < 1$; hence the inertia-free approximation is more valid for the right than the left side.

the inner layer, on the solution. Figure 3 shows the dependence of the growth rate of the matching scheme solution on $\Delta r$. The growth rate is normalized to $\gamma_{\mathrm{global}}$, the growth rate from the global solution. The growth rate agrees well with $\gamma_{\mathrm{global}}$ for $\Delta r \geq 0.2$. When $\Delta r \leq 0.1$, we have $\gamma > \gamma_{\mathrm{global}}$; it is understood that the matching scheme underestimates the kinetic energy for the solution. When $\Delta r$ is chosen to be less than 0.05, we did not obtain a solution that grew from the initial noise level within the simulation time, $\omega_{\mathrm{pa}} t = 400$. Figure 4 shows the matching scheme error of $y(r)$ from the global solution as a function of $r$. The solid, broken, and dotted lines represent $\Delta r = 0.3$, 0.2, and 0.1, respectively. Notice that maximum in $y(r)$ is unity. We observe that the agreement between $y(r)$ and the global solution is already good for $\Delta r = 0.1$. We also observe that the error profile is unsymmetrical with respect to $r = 0.4$ — the position of $q(r) = 1$. The magnetic shear is stronger on the right side of $q > 1$ than the left side of $q < 1$; hence the inertia-free approximation is more valid on the right side than the left side. We can conclude that these results verify the present scheme for MHD stability analysis.

In summary, we have proposed a new matching scheme for MHD stability problems that is tractable from a numerical viewpoint. We have presented the formulation and the numerical tests using the linear ideal MHD equation. The new scheme provides the matching conditions in the form of the linear equation, which can be solved easily, on the values of the radial displacement at the matching points. We emphasize that the implicit scheme of the evolution equation is crucial in deriving the matching conditions. We also emphasize that the new scheme in the example of $\Delta r = 0.1$ in Fig. 1 reduces the CPU time to one tenth of that required when the equation is solved for the full range.

On adopting the present scheme, we can flexibly change the physical model from the linear ideal MHD equation to more complex MHD equations, around an arbitrarily chosen rational surface. We can easily clarify the new effects on MHD stability at that rational surface. One of such MHD equations is the Frieman-Rotenberg equation that describes ideal MHD motion in a rotating plasma [16, 17]. This equation has recently received attention in the theory of resistive wall modes [18]. Another interesting subject is the application to resistive MHD modes. While for ideal modes the layer width is determined by the growth rate and therefore so-called *a priori* estimation of the width is difficult, for resistive MHD modes it can be estimated in advance by the electrical resistivity at the rational surface. The new scheme is expected to substantially reduce the CPU time required for such problems. Applications to those problems will be reported in the near future.

[1] H.P. Furth, J. Killeen and M.N. Rosenbluth, Phys. Fluids **6**, 459 (1963).
[2] R.D. Hazeltine and J.D. Meiss, *Plasma Confinement* (Addison-Wesley, Redwood, 1992).
[3] W.A. Newcomb, Ann. Phys. (N.Y.) **10**, 232 (1960).

[4] A.H. Glasser, J.M. Greene and J.L. Johnson, Phys. Fluids **18**, 875 (1975).
[5] A.H. Glasser, J.M. Greene and J.L. Johnson, Phys. Fluids **19**, 875 (1976).
[6] A. Pletzer and R.L. Dewar, J. Plasma Phys. **45**, 427 (1991).
[7] S. Tokuda, Nucl. Fusion **41**, 1037 (2001).
[8] A.H. Glasser, S.C. Jardin and G. Tesauro, Phys. Fluids **27**, 1225 (1984).
[9] S. Tokuda, J. Plasma Fusion Res. **77**, 276 (2001).
[10] S. Tokuda and T. Watanabe, J. Plasma Fusion Res. **73**, 1141 (1997).
[11] A. Pletzer, A. Bondeson and R.L. Dewar, J. Comput. Phys. **115**, 530 (1994).
[12] S. Tokuda and T. Watanabe, Phys. Plasmas **6**, 3012 (1999).
[13] A.H. Glasser, Los. Alamos Report LA-UR-95-528 (1997).
[14] I.B. Bernstein, E.A. Frieman, M.D. Kruskal and R.M. Kulsrud, Proc. R. Soc. **A244**, 17 (1958).
[15] R. Gruber and J. Rappaz, *Finite Elements Methods in Linear Ideal Magnetohydrodynamics* (Springer, Berlin, 1985).
[16] E. Frieman and M. Rotenberg, Rev. Mod. Phys. **32**, 898 (1960).
[17] S. Tokuda, J. Plasma Fusion Res. **74**, 503 (1998).
[18] L.-J. Zheng, M. Kotschenreuther and M.S. Chu, Phys. Rev. Lett. **95**, 25003 (2005).