Spectral Effect of Zonal Flows and Enhanced Growth Rate

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The effect of the spectrum of the radial wave number of zonal flows on zonal flow generation is theoretically investigated using the Hasegawa-Mima turbulence model by representing the spectrum by means of two monochromatic waves. Based on this method, we explored a ten-wave coupling model of modulational instability. We found that the zonal flow generation is qualitatively different in cases with and without such a spectral effect, exhibiting the enhancement of the growth rate. This originates from the coupling property of the new zonal flow eigenmode equation system. We refer to this state as a *coupled zonal flows eigenmode*, which leads to a spatial modulation of zonal flows affected by the turbulence structure.

Keywords: drift wave turbulence, zonal flow, modulational instability, electron temperature gradient (ETG) mode

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It is well recognized that zonal flows, which are nonlinearly generated from micro-scale turbulence, play an important role in regulating turbulence and transport in magnetically confined fusion plasmas [1, 2]. For understanding such a turbulence-zonal flow system, it is necessary to elucidate the spatio-temporal structures of zonal flows and turbulence and their mutual relation. The generation processes of zonal flows have been intensively studied based on the modulational instability. For studying such a modulation process, two approaches based on the Hasegawa-Mima (HM) model, namely, the coherent mode coupling (CMC) approach [3–5] and the wave kinetic (WK) approach [6–8], have been developed. In these methods, zonal flows have been modeled by a monochromatic wave with a sinusoidal function. However, the generation of zonal flows is purely a nonlinear process, which essentially involves complex coupling among spectral distributions of zonal flows and turbulence.

To address this problem, we present a model that includes the effect of the spectrum of the radial wave number of zonal flows and turbulence by means of two monochromatic waves with a difference of δk_x, which are given by radial and poloidal wave numbers (k_{x0}, k_y) and $(k_{x1} \equiv k_{x0} + \delta k_x, k_y)$ for pump waves, and $(k_{q0}, 0)$ and $(k_{q1} \equiv k_{q0} + \delta k_x, 0)$ for zonal flows. This is considered to be the minimum model, which represents the qualitatively different aspects of zonal flow generation compared with the monochromatic treatment. Then, six side-bands, i.e., $(k_{xj\pm} \equiv k_{x0} + j\delta k_x \pm k_{q0}, k_y)$ with $j = 0$, 1, and 2, can be produced through a nonlinear interaction. Therefore, the present system consists of ten waves, and is equivalent to that involving two sets of four-wave coupling loops with a spectral shift δk_x, i.e., Loop A: $\{(k_{x1}, k_y),\ (k_{q0}, 0)$ and $(k_{x1+}, k_y)\}$ and Loop B: $\{(k_{x0}, k_y),\ (k_{q1}, 0)$ and $(k_{x1+}, k_y)\}$. Although, this system is not a simple superposition of two sets, a new cross coupling appears between these two sets through the side-band with the wave numbers (k_{x1+}, k_y), which links two pairs of modulation loops. This cross coupling results in an interaction between the two loops, and then influences the zonal flow instability. Following this idea, we derive a dispersion relation of zonal flow instability as follows.

Expanding each potential field $\phi(\boldsymbol{r}, t)$ as $\phi(\boldsymbol{r}, t) = \frac{1}{2}\sum_{\boldsymbol{k}}[\phi_{\boldsymbol{k}}(t) \exp i\boldsymbol{k}\cdot\boldsymbol{r} + \text{c.c.}]$, the HM equation modeling the electron temperature gradient (ETG) turbulence and zonal flow system reads to

$$\frac{\mathrm{d}\phi_{\boldsymbol{k}}}{\mathrm{d}t} + i\omega_{\boldsymbol{k}}\phi_{\boldsymbol{k}} = \sum_{\boldsymbol{k}=\boldsymbol{k}'+\boldsymbol{k}''} \Lambda^{\boldsymbol{k}}_{\boldsymbol{k}',\,\boldsymbol{k}''}\phi_{\boldsymbol{k}'}\phi_{\boldsymbol{k}''}, \tag{1}$$

where $\omega_{\boldsymbol{k}}$ is the drift wave frequency (note that $\omega_{\boldsymbol{k}} = 0$ for zonal flow). The normalization in Eq. (1) is the same as Ref. [4]. The matrix element $\Lambda^{\boldsymbol{k}}_{\boldsymbol{k}',\,\boldsymbol{k}''}$ is given by $\Lambda^{\boldsymbol{k}}_{\boldsymbol{k}',\,\boldsymbol{k}''} = \boldsymbol{k}' \times \boldsymbol{k}'' \cdot \hat{z}(k''^2 - k'^2)/[2(1 + k^2)]$. The pump waves are expressed as $\tilde{\phi}^{(p)}(\boldsymbol{r}, t) = \sum_{j=0}^{1} \phi_{k_{xj},k_y}(t) \exp i\,(k_{xj}x + k_y y + \theta_j) +$ c.c. An arbitrarily small perturbation is chosen as a seed of the zonal flows with two wave components $\tilde{\phi}^{(q)}(\boldsymbol{r}, t) = \sum_{j=0}^{1} \phi_{k_{qj},0}(t) \exp ik_{qj}x + \text{c.c.}$ Corresponding side-bands are $\tilde{\phi}^{(s\pm)}_{\boldsymbol{k}}(\boldsymbol{r}, t) = \sum_{j=0}^{2} \phi_{k_{xj\pm},k_y}(t) \exp i\,(k_{xj\pm}x + k_y y + \theta_j) + \text{c.c.}$, where θ_j indicates initial phase factors. Then, we divide all fields into a slowly varying envelope part and a rapidly varying eikonal as $\phi_i(t) = A_i(t) \exp(-i\omega_i t + i\theta_j)$, which represent two zonal flows for $i = (1, 1')$, two pump waves for $i = (2, 2')$, and two pairs of side-bands for $i = (3, 3', 3'')$ and $(4, 4', 4'')$, respectively. Note that ω_i represents real frequencies. Assuming that the two pump waves, $|A_2|$ and $|A'_2|$, have larger amplitudes than that of other waves, we assume these amplitude to be same and constant as $|A_2| = |A'_2|$. We also assume that the real frequency of zonal flow is small enough compared with drift frequency [5];

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therefore, $\omega_1 = \omega'_1 = 0$ is chosen. Then, we obtain the following two equations for zonal flows in matrix form:

$$\frac{d^2}{dt^2}\begin{pmatrix} A_1 \\ A'_1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A'_1 \end{pmatrix}, \quad (2)$$

where the matrix coefficients are given by $a_1 = (\Lambda^1_{2^*,3}\Lambda^3_{1,2} + \Lambda^1_{2,4}\Lambda^4_{1,2^*} + \Lambda^1_{2'^*,3''}\Lambda^{3''}_{1',2'} + \Lambda^1_{2',4'}\Lambda^{4'}_{1,2'^*})|A_2|^2$, $a_2 = (\Lambda^{1'}_{2^*,3'}\Lambda^{3'}_{1,2'} + \Lambda^{1'}_{2',4}\Lambda^4_{1,2^*})|A_2|^2$, $b_1 = (\Lambda^1_{2'^*,3'}\Lambda^{3'}_{1',2} + \Lambda^1_{2,4}\Lambda^4_{1',2'^*})|A_2|^2$, and $b_2 = (\Lambda^{1'}_{2^*,3'}\Lambda^{3'}_{1',2} + \Lambda^{1'}_{2',4}\Lambda^4_{1',2'^*} + \Lambda^{1'}_{2',3''}\Lambda^{3''}_{1',2'} + \Lambda^{1'}_{2,4''}\Lambda^{4''}_{1',2^*})|A_2|^2$. Note that the phase difference $\Delta\theta_j \equiv |\theta_1 - \theta_0|$ does not appear in Eq. (2) explicitly; therefore, there is no effect of the initial phases on the zonal flow growth rate. Here, we find a solution in which the two zonal flows and six side-bands have the same growth (or damping) rate γ_q, which corresponds to an *eigenmode* in which all eight waves couple each other. By setting $A_i(t) = A_i^{(0)}(t)\exp(\gamma_q t)$, which excludes two pump waves for $i = (2, 2')$, Eq. (2) yields a fourth-order algebraic equation with respect to γ_q expressed as

$$\gamma_q^4 - (a_1 + b_2)\gamma_q^2 + (a_1 b_2 - a_2 b_1) = 0. \quad (3)$$

First, Eq. (3) is solved and compared with the growth rate based on the four-wave model, where waves with $i = (1', 2', 3', 3'', 4', 4'')$ in Eq. (2) are excluded. The spectral difference $\delta k_x = 0.0125$ is chosen. Since the generation of zonal flows depends on the pump amplitude, we choose the same pump energy in both models for direct comparison of the growth rates. Therefore, the amplitude of each pump component in the present ten-wave model is reduced by a factor $1/\sqrt{2}$ to keep the total pump wave energy the same as that of the four-wave model. In Fig. 1 (a), the growth rate of the zonal flow γ_q is plotted with respect to the radial wave number of the pump wave k_{x0} for the two models. It is found that the growth rate of zonal flow in the ten-wave model $\gamma_q^{(10)}$ is larger compared with that in the four-wave one $\gamma_q^{(4)}$ in a wide k_{x0} region. Next, we examine the dependence of the growth rate on the spectral difference δk_x. Here, we solved Eq. (3) from $\delta k_x = 0.1$ down to 10^{-4} as shown in Fig. 1 (b). It is seen that the growth

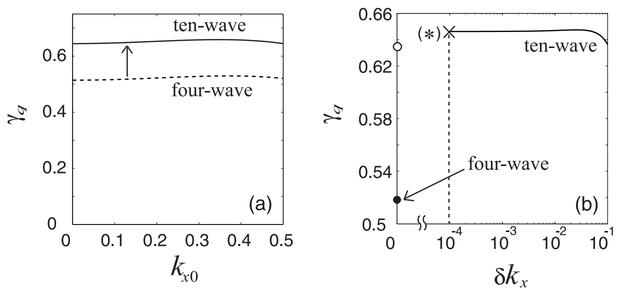

Fig. 1 Growth rate γ_q of zonal flow as a function of (a) the wave number k_{x0} and (b) the difference between two radial wave numbers of turbulence dk_x. Open circle (∘) at $\delta k_x = 0$ represents $\sqrt{3/2}\gamma_q^{(4)} \simeq 0.637$.

rate γ_q weakly depends on δk_x, keeping an almost constant value down to $\delta k_x = 10^{-4}$ at which $\gamma_q = 0.645$ is estimated (marked by (∗)). Since the growth rate in the four-wave model is $\gamma_q \simeq 0.52$, Fig. 1 (b) suggests a difference (and/or discontinuity) between two cases of around $\delta k_x = 0$ with and without the spectral nature of zonal flow.

In order to precisely identify the dynamics around $\delta k_x = 0$, we analyze the dispersion relation Eq. (2) in the small δk_x region. Expanding each coefficient with respect to δk_x in Eq. (2) and keeping the first order, we obtain $a_1 = \gamma_q^{(4)} + C_1(k_{x0}, k_y, k_{q0})\delta k_x$, $a_2 = \gamma_q^{(4)}/2 + C_2(k_{x0}, k_y, k_{q0})\delta k_x$, $b_1 = \gamma_q^{(4)}/2 + C_3(k_{x0}, k_y, k_{q0})\,\delta k_x$, and $b_2 = \gamma_q^{(4)} + C_4(k_{x0}, k_y, k_{q0})\,\delta k_x$, where coefficients C_1 to C_4 are functions of (k_{x0}, k_y) and k_{q0} with no specific singularities. Then, the solution up to the first order with respect to δk_x of Eq. (3) yields

$$\gamma_q = \sqrt{3/2}\gamma_q^{(4)}\left[1 + C(k_{x0}, k_y, k_{q0})\delta k_x\right], \quad (4)$$

where C is a positive regular function of (k_{x0}, k_y) and k_q. Equation (2) is valid when δk_x is not zero but finite (i.e., $\delta k_x > 0$), since the two coupling pairs are assumed *a priori* in the dispersion relation in the present ten-wave model. However, it is found that Eq. (4) can be analytically connected to $\delta k_x = 0$ as found from Eq. (4); therefore, the difference (and/or discontinuity) exists between $\gamma_q^{(10)}$ and $\gamma_q^{(4)}$ at small δk_x by $\sqrt{3/2}$ as plotted in Fig. 1 (b), suggesting that the present ten-wave model provides a qualitatively different characteristic due to the spectral nature of zonal flows. Note that if only monochromatic zonal flow is considered in the coupling system, there is no essential difference in the growth rate γ_q between the two cases, whether or not the spectral nature of pump wave is considered.

In conclusion, we found that the zonal flow growth rate is qualitatively different in cases with and without the spectrum effect of zonal flows. This originates from the coupling property of the new zonal flow eigenmode equation system, which leads to a discontinuity compared with the four-wave model at small δk_x limit. We refer to this state as a *coupled zonal flows eigenmode*. This process is similar in form to the toroidal eigenmode, where different poloidal harmonics couple each other through toroidicity, causing a ballooning mode. However, it is noted that the present eigenmode is determined for a given turbulent spectrum. Here, we have shown a simple case, where the radial spectra of zonal flows and turbulence are modeled by two monochromatic waves. However, to predict the zonal flow growth rate quantitatively, a direct numerical simulation considers precise spectral information of zonal flows and turbulence including the phases is necessary. Meanwhile, Gaussian and/or wider turbulence and zonal flow spectrum may cause more complex interactions, leading to a spatially localized wave packet, which will be discussed in a future publication.

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