Symmetries and solutions of equations describing force-free magnetic fields

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New analytical results for two-dimensional force-free fields are presented. First, a number of exact solutions for force-free fields described by Liouville equation are given and their physical relevance for laboratory and astrophysical plasmas is discussed. Subsequently, Lie point symmetries of Liouville equation are reviewed and Lie point symmetries of the equation describing helically symmetric force-free fields are presented and discussed. A formal analogy between the equations governing the vector potential components of a class of force-free fields and the dynamical equations for a charged particle in a magnetic field is also shown. Examples of solutions and applications of this analogy are discussed.

Keywords: force-free magnetic fields, Liouville equation, Lie point symmetries, exact solutions

1. Introduction

In many strongly magnetized plasmas stationary magnetic structures can be adequately described by means of force-free (FF) magnetic fields [1]. Such fields are characterized by a vanishing Lorentz force, i.e., in addition to the divergence-free condition, they also satisfy the relation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = 0, \tag{1}$$

where $\mathbf{B}$ is the magnetic vector field. FF magnetic fields are commonly adopted in order to model, e.g., coronal structures (see, e.g. [2, 3, 4]) or, according to Taylor's classical theory [5, 6], relaxed states in Reversed Field Pinches (RFP). In spite of their wide range of applicability, not many examples of exact solutions for FF fields are known. Most of these (e.g. [7, 8, 9, 10] refer to the case in which an ignorable spatial coordinate is present and, even in this simplified case, the search for exact solutions of (1) is a very complex problem, especially when the resulting equation is nonlinear. In Sec. 2 of this contribution we present a nonlinear case for which, remarkably, an infinite variety of exact solutions is available. The case under consideration is the one in which the FF condition (1) corresponds to Liouville equation, whose general solution is known [11]. A number of solutions of this equation, in particular, can be of interest in order to model two-dimensional ($2D$) magnetic structures present in laboratory and astrophysical plasmas. This richness of solutions can be given an explanation also in terms of the infinite-dimensional algebra of Lie point symmetries of Liouville equation [12, 13, 14].
A case of FF fields that is not covered by Liouville equation is that of FF helically symmetric fields. These fields can be of practical interest, e.g., for modelling sigmoidal structures in the solar corona [15]. Existence of helically symmetric equilibrium solutions could also provide support to the argument, according to which, Quasi-Single-Helicity states observed in RFP can be seen as an "attempt" by the plasma to reach a laminar, helically symmetric equilibrium [16].
In Sec. 3 we present an investigation of Lie point symmetries of the equation describing helically symmetric FF fields and discuss the results in terms of the possibility of finding new solutions.
In Sec. 4 we point out that the equations for the vector potentials of $1\frac{1}{2}D$ FF fields in Cartesian coordinates are formally analogous to those governing the motion of a charged particle in a magnetic field. This analogy helps us to gain knowledge about FF fields, by transferring what is known from the classical problem of the motion of a charged particle in an inhomogeneous magnetic field. Sec. 5 is devoted to conclusions.

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2. Exact solutions for $2D$ nonlinear force-free fields

Let us consider a Cartesian coordinate system (x, y, z) and a magnetic field $\mathbf{B}$ not depending on the variable z. Such a vector field can be written as

$$\mathbf{B} = \nabla\psi(x, y) \times \mathbf{e}_z + B_z(x, y)\mathbf{e}_z, \tag{2}$$

where ψ is denoted as a flux function and B_z is the amplitude of the field along the z direction, identified by the unit vector $\mathbf{e}_z$. For this type of magnetic fields, B_z turns out to be a function of ψ only and the FF condition (1) reduces to the scalar equation

$$\nabla^2\psi + \left(\frac{B_z^2}{2}\right)' = 0, \tag{3}$$

where B_z is a free function and $'$ denotes derivative with respect to the argument. Different choices for the dependence of B_z on ψ lead then to different classes of solutions for FF equilibria. In particular, if B_z is a linear function of ψ, the resulting magnetic fields are referred to as linear FF field. A particularly relevant choice is given by $B_z(\psi) = \exp(\psi)$, which yields the Liouville equation

$$\nabla^2\psi = -\exp(2\psi). \quad (4)$$

This choice is remarkable because the general solution of Liouville equation is known [11]. Indeed it can be shown that the general solution of (4) is given by

$$\psi(x,y) = \ln\left(\frac{2|g'(\zeta)|}{1+|g(\zeta)|^2}\right), \quad (5)$$

where $g(\zeta)$ is an arbitrary holomorphic function of the complex variable $\zeta = x + iy$. The corresponding solution for B_z will then, of course, be given by

$$B_z(x,y) = \frac{2|g'(\zeta)|}{1+|g(\zeta)|^2}. \quad (6)$$

Notice also that, in this case, the associated current density $\mathbf{j}$ is given by $\mathbf{j} = B_z\mathbf{B}$.
From the expression (5) it emerges then that every holomorphic function yields a solution of Liouville equation and, consequently a solution for a FF field. Notice that Liouville equation governs also isothermal equilibria with a constant B_z. Solutions in this context have been presented and discussed in [17, 18, 19]. In the following we present physically relevant analytical solutions for FF magnetic fields corresponding to different choices of $g(\zeta)$.
A first example consists of choosing $g(\zeta) = \kappa\zeta$, with constant κ. The resulting field, when expressed in cylindrical coordinates (r, θ, z), reads

$$B_r = 0, \qquad B_\theta = \frac{2\kappa^2 r}{1+(\kappa r)^2},$$
$$B_z = \frac{2\kappa}{1+(\kappa r)^2},$$

where $r^2 = x^2 + y^2$. This corresponds to the well known Gold-Hoyle field [20], which is adopted in order to describe twisted flux tubes emerging from the photosphere. The Gold-Hoyle field is characterized by possessing no magnetic shear, in the sense that the twist, with which the field lines wind around the flux tube, is independent of the radius. In the above mentioned context of isothermal equilibria the counterpart of the Gold-Hoyle field is the Bennet pinch [21]
A further classical solution covered by Liouville equation is the FF Harris sheet [22], whose expression reads

$$B_x = 0, \qquad B_y = \tanh x, \qquad B_z = \mathrm{sech} x. \quad (7)$$

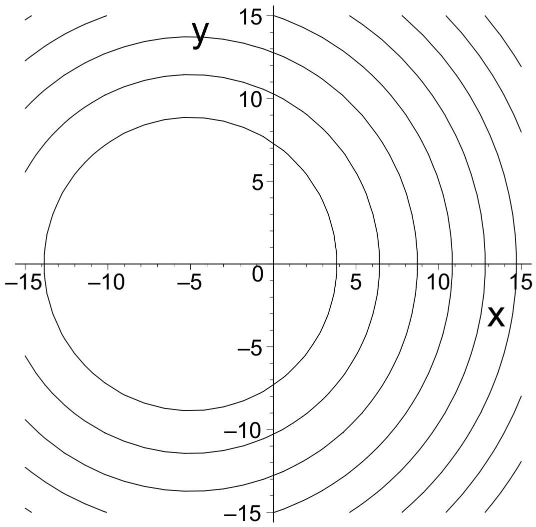

Fig. 1 Contour lines in the poloidal plane of the solution (8)-(9) for $p = 0.2$.

This represents a paradigmatic equilibrium for studying reconnecting instabilities. The classical Gold-Hoyle and Harris solutions can be seen as special cases of a family of solutions corresponding to take $g(\zeta) = (1 + p\zeta)^{1/p}$ with $p > 0$. The resulting field components of this family of solutions are given by

$$B_\theta = \frac{p\bar{r}^{\frac{4}{p}} + 2p\bar{r}^{\frac{2}{p}} + p + \bar{r}^{\frac{4}{p}} - 1}{p\bar{r}(\bar{r}^{\frac{2}{p}}+1)^2}, \quad (8)$$

$$B_r = 0, \qquad B_z = \frac{2}{\bar{r}(\bar{r}^{\frac{1}{p}} + \bar{r}^{-\frac{1}{p}})}, \quad (9)$$

where $\bar{r} = [(1+px)^2 + p^2y^2]^{1/2}$. These describe circular field lines in the poloidal (i.e. $z = 0$) plane, centered at $(x = -1/p, y = 0)$ and of radius $\bar{r}$. At $p = 0$ the center of the circles is at $-\infty$ so that the field lines in the poloidal plane consist of straight lines and the Harris sheet is recovered. For $0 < p < 1$ the B_z component has a maximum at finite $\bar{r}$ but then decays to zero both at $\bar{r} = 0$ and at $\bar{r} \to \infty$. For $p = 1$ a Gold-Hoyle solution is recovered, whereas for $p > 1$ the B_z component has a singularity at $\bar{r} = 0$ and goes to zero for $\bar{r} \to \infty$. Therefore, in general, this family of solutions can be used to model flux tubes with magnetic shear. An example of the poloidal contour lines for this type of solutions is provided in Fig. 1.
A $2D$ family of solutions that extends the Harris sheet solution can be found by choosing $g(\zeta) = \sqrt{1+a^2}\exp(b\zeta) + a$, with a, b constant and $b > 0$. This represents another classical type of solution, which in the fluid dynamics context is known as Kelvin-Stuart "cat's eye" solution. In the plasma context this solution describes a sequence of magnetic islands in the poloidal plane (often adopted to study magnetic island coalescence [23, 24, 25, 26]), whereas B_z peaks at the O-points of the islands and decays to zero at

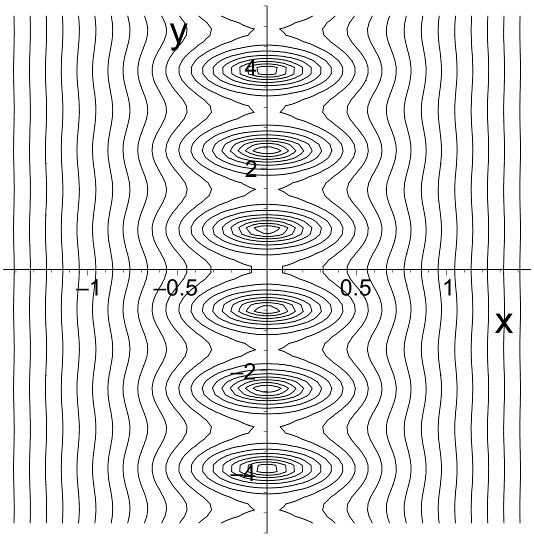

Fig. 2 Contour lines in the poloidal plane of the solution given by $g(\zeta) = \sqrt{1+a^2}\exp(b\zeta) + a$ for $a = 2$, $b = 4$.

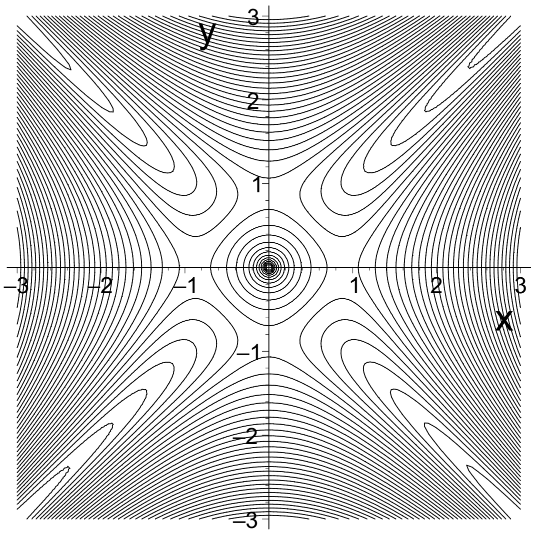

Fig. 3 Contour lines in the poloidal plane of the solution given by $g(\zeta) = \exp(\zeta^2)$.

large x. A plot of the magnetic island structure in the poloidal plane is shown in Fig. 2.

Another interesting solution can be obtained with the choice $g(\zeta) = \exp(\zeta^2)$.

This solution, whose analytical expression for the flux function is given by

$$\psi(x, y) = -\ln\left[\frac{\cosh(x^2 - y^2)}{2\sqrt{x^2 + y^2}}\right] \qquad (10)$$

describes, in the poloidal plane, as can be seen in Fig. 3, a quadrupolar structure. The magnetic field possesses four X-points in the $z = 0$ plane and a singularity at the origin. B_z, on the other hand, has no singularity and is concentrated along the four lobes of the quadrupolar structure.

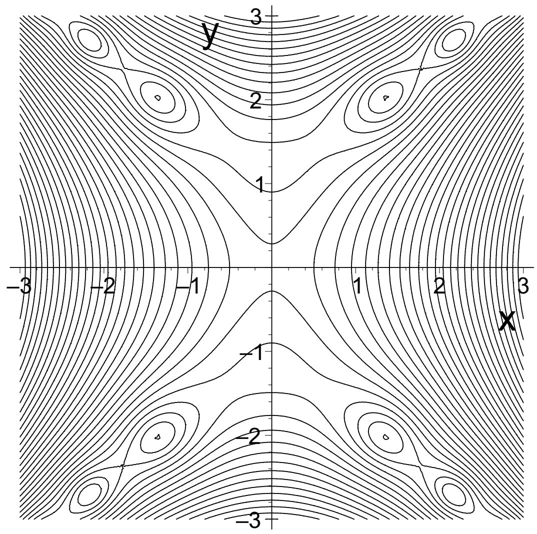

Fig. 4 Contour lines in the poloidal plane of the solution given by $g(\zeta) = p\,\mathrm{erf}(\zeta)$ for $p = 0.5$.

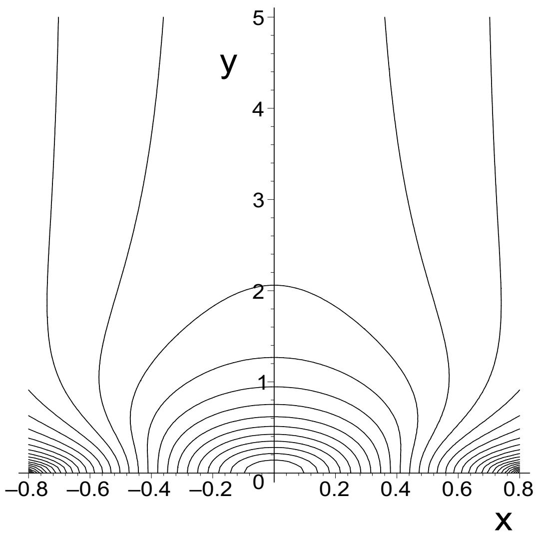

Fig. 5 Contour lines in the poloidal plane of the solution given by $g(\zeta) = \exp(\zeta)(\zeta + p)/(\zeta - p)$ for $p = 0.3$.

Further nonlinear solutions of interest are obtained with $g(\zeta) = p\,\mathrm{erf}(\zeta)$, where $p > 0$ and erf is the error function. The resulting magnetic configuration (see Fig. 4) is characterized by possessing an elongated current layer along the y axis. Four sequences of magnetic islands are present, in correspondence to the four lobes emanating from the separatrices of the current layer.

Finally, a solution that can be of use to model arcade-like structures emerging from the photosphere, is obtained by choosing $g(\zeta) = \exp(\zeta)(\zeta + p)/(\zeta - p)$ with constant p. A plot of the poloidal contour lines is shown in Fig. 5. Note that also this family of solutions includes the Harris sheet as a special case, namely when $p = 0$ or $p \to \infty$. In the former case the two singularities present at $y = 0$ merge at the origin whereas in the latter case they move to infinity.

3. Lie point symmetries of equations describing force-free fields with translational and helical symmetry

The technique of Lie point symmetries [27] provides a systematic way for deriving exact solutions of a given differential equation. In particular, the knowledge of the group of symmetries of a differential equation can be applied, for instance, to find one-parameter groups of solutions from a known solution, or to find the solutions which are invariant under a given symmetry.

Symmetries of the Liouville equation have been discussed in Refs. [12, 13, 14]. Here we limit ourselves to recall that, for each holomorphic function $h(\zeta) = u(\zeta) + iv(\zeta)$, the vector field (written according to the standard notation adopted, e.g. in [27])

$$X = u(x,y)\frac{\partial}{\partial x} + v(x,y)\frac{\partial}{\partial y} - \frac{\partial u}{\partial x}\frac{\partial}{\partial \psi} \tag{11}$$

is a generator of a group of symmetry of the Liouville equation. Therefore the mapping

$$\begin{aligned} &\tilde{x} = \tilde{x}(x,y,\lambda), \qquad \tilde{y} = \tilde{y}(x,y,\lambda), \\ &\tilde{\psi} = \tilde{\psi}(\psi,x,y,\lambda), \end{aligned} \tag{12}$$

which corresponds to the solution of the system

$$\frac{dx}{d\lambda} = u, \qquad \frac{dy}{d\lambda} = v, \qquad \frac{d\psi}{d\lambda} = -\frac{\partial u}{\partial x}, \tag{13}$$

with initial conditions $\tilde{x}(x,y,0) = x$, $\tilde{y}(x,y,0) = y$, $\tilde{\psi}(\psi,x,y,0) = \psi$, provides a group of transformations, depending on the parameter λ, of the dependent and independent variables, that makes it possible to construct a family of solutions from a known one. The Lie symmetries of Liouville equation are also related to the general solution (5) by the following property. If X is a vector field of the form (11), associated to a holomorphic function $h = u + iv$ and $\tilde{u}(x,y)$ is a solution of (4), associated, through (5), to a holomorphic function $g(\zeta)$, then the functions $h(\zeta)$ and $g(\zeta)$ are related by

$$g'(\zeta)h(\zeta) = ig(\zeta). \tag{14}$$

For instance, in the case of the "cat's eye" solution with $b = 1$, it can be verified that the corresponding solution for the flux function is left invariant by

$$\begin{aligned} X = &\frac{a}{\sqrt{1+a^2}}\exp(-x)\sin y\frac{\partial}{\partial x} \\ &+ \left(1 + \frac{a}{\sqrt{1+a^2}}\exp(-x)\cos y\right)\frac{\partial}{\partial y} \\ &+ \frac{a}{\sqrt{1+a^2}}\exp(-x)\sin y\frac{\partial}{\partial \psi}. \end{aligned} \tag{15}$$

This vector field is associated to the holomorphic function $h(\zeta) = i(\sqrt{1+a^2}\exp(z) + a)/\sqrt{1+a^2}\exp(z)$, which satisfies (14), for $g(\zeta) = \sqrt{1+a^2}\exp(\zeta) + a$. Further symmetry properties of Liouville equation have been also discussed in [18, 19].

Another relevant class of FF fields consists of those which possess helical symmetry, i.e., in cylindrical coordinates, which are functions only of the radius and of the helical coordinate $u = \theta + kz$, with constant k. Such fields can be conveniently written in the form

$$\mathbf{B}(r,u) = \nabla\chi(r,u) \times \mathbf{h} + g(r,u)\mathbf{h}, \tag{16}$$

where $\mathbf{h} = (r/(1+k^2r^2))\nabla r \times \nabla u$ identifies a helical direction, $\chi(r,u)$ is the helical flux function and $g(r,u)$ is the amplitude of $\mathbf{B}$ along $\mathbf{h}$. The FF condition (1) for such fields corresponds to the equation

$$\begin{aligned} &\frac{\partial^2\chi}{\partial r^2} + \frac{1-k^2r^2}{1+k^2r^2}\frac{\partial\chi}{\partial r} + \frac{1+k^2r^2}{r^2}\frac{\partial^2\chi}{\partial u^2} + \\ &\frac{2k}{1+k^2r^2}g + \left(\frac{g^2}{2}\right)' = 0. \end{aligned} \tag{17}$$

Similarly to B_z in the $2D$ case treated in Sec. 1, $g(\chi)$ here is a free flux function. Indeed the $2D$ case with translational symmetry along z, is retrieved by considering the $k = 0$ limit of (17). Investigation of the Lie symmetries of Eq. (17) has been performed with the help of symbolic computation software (see, e.g. [28]). This investigation showed that, for any g, Eq. 17 possesses only the symmetry generated by the vector field $X = \partial/\partial u$, which merely implies that if $\chi(r,u)$ is a solution of (17), then $\chi(r, u+\lambda)$, with constant λ, is also a solution. This indicates that interesting symmetries for helically symmetric magnetic fields could exist only for particular choices of $g(\chi)$. In particular we found out that the choice $g(\chi) = \exp(\alpha\chi)$, with constant α, does not add further symmetries, unless $k = 0$ (which brings back to Liouville equation). The choice $g = \mu\chi$, with constant μ, which was treated in Ref.[29] and which leads to Taylor helical states [5, 6], only adds the symmetry following from the linearity of the equation. Nevertheless we found out that, in the limit of $kr \ll 1$, retaining only first order terms in kr, Eq. (17) acquires, for *any* $g(\chi)$, the symmetry generated by

$$X = \sin u\frac{\partial}{\partial r} + \frac{1}{r}\cos u\frac{\partial}{\partial u}, \tag{18}$$

which produces the transformation

$$\begin{aligned} &r \to \sqrt{r^2 - 2\lambda r\sin u + \lambda^2}, \\ &\cos u \to \frac{r\cos u}{\sqrt{r^2 - 2\lambda r\sin u + \lambda^2}}. \end{aligned} \tag{19}$$

Therefore, thanks to this property, in the vicinity of the cylinder axis, it is possible to construct helically symmetric solutions from purely radial solutions. For instance one could obtain a helically symmetric solution from the Bessel Function Model solution $\chi(r) = J_0(\mu r) - krJ_1(\mu r)$, adopted by Taylor to model relaxed magnetic states in RFP.

4. Analogy between force-free fields and charged particles in a magnetic field

Let us consider planar FF fields of the form

$$\mathbf{B} = B_y(x)\mathbf{e}_y + B_z(x)\mathbf{e}_z. \tag{20}$$

The FF condition implies that the corresponding vector potential components $A_y(x)$, $A_z(x)$ satisfy

$$\frac{d^2 A_i}{dx^2} = \lambda(x)\varepsilon_{xij}\frac{dA_j}{dx}, \qquad i,j = y,z \tag{21}$$

where ε_{ijk} is the Levi-Civita tensor and $\lambda(x)$ is a function such that $\mathbf{j} = \lambda(x)\mathbf{B}$. An alternative expression for (21) is

$$\frac{d^2 A_i}{dx^2} = \Lambda(A_y, A_z)\varepsilon_{xij}\frac{dA_j}{dx}, \qquad i,j = y,z. \tag{22}$$

Eqs. (22) and (21) can be formally interpreted as the equations of motion for a pseudo-particle of unit mass and charge, with pseudo-coordinates A_y, A_z subject to the presence of a pseudo-magnetic field directed along $\mathbf{e}_{A_x}$. In the case of (22) the amplitude of the magnetic field depends on the two spatial pseudo-coordinates (both functions of x), whereas in (21) the amplitude λ depends explicitly on the pseudo-time x. Within this analogy, the kinetic energy of the pseudo-particle, which is given by

$$H = \left(\frac{dA_y}{dx}\right)^2 + \left(\frac{dA_z}{dx}\right)^2 = |\mathbf{B}|^2 = \text{constant},$$

corresponds to the magnetic energy of the FF field and is a conserved quantity. Note that this analogy fails if one considers $1\frac{1}{2}D$ FF fields in coordinates other than Cartesian. Indeed, in general coordinates, a magnetic tension term is also present, which prevents the magnetic energy from being constant and breaks the analogy with the Hamiltonian system describing the dynamics of the pseudo-particle. The pseudo-angular momentum of the pseudo-particle, on the other hand, is given by

$$M_{A_x} = A_y\frac{dA_z}{dx} - A_z\frac{dA_y}{dx} = -\mathbf{A}\cdot\mathbf{B}, \tag{23}$$

and is proportional to the magnetic helicity density $\mathbf{A}\cdot\mathbf{B}$. Such quantity is of course conserved when the Lagrangian of the pseudo-particle is invariant with respect to rotations.
The above described analogy can be of interest for it allows us to transfer known results from the theory of the motion of a charged particle in a magnetic field, into the context of FF fields. In the remainder of this section we list a few examples of this analogy, that correspond to different choices of the functions Λ.

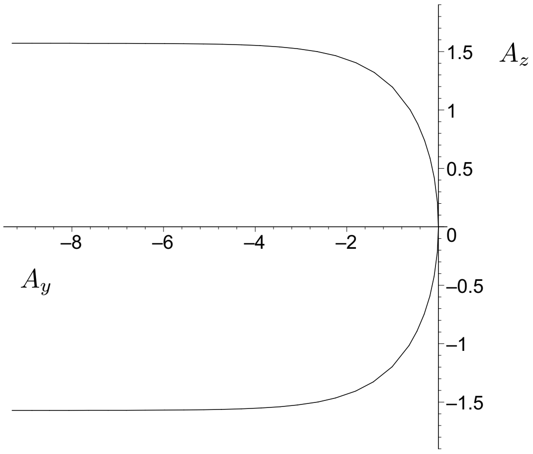

Fig. 6 Trajectory of a pseudo-particle in a $1D$ pseudo-configuration corresponding to the FF Harris sheet.

4.1 $0D$ pseudo-configurations

These correspond to the choice Λ = constant, which means that the pseudo-particle is subject to a uniform, time-independent magnetic field. The trajectories of the pseudo-particle consist then of circular orbits and the corresponding FF field lines consist of straight lines whose orientation in the B_yB_z plane plane rotates with constant speed as one moves along the x axis. In this case the Lagrangian of the pseudo-particle is invariant under rotation so the pseudo-angular momentum is conserved. The magnetic helicity density of the FF counterpart is indeed uniform. Note that for this type of FF field (and, to the best of our knowledge, only for this), a corresponding solution of the stationary Vlasov equation has been found [30, 31].

4.2 $1D$ pseudo-configurations

In this case Λ is a function of only one variable, say A_z. Denoting with $\mathcal{A} = \mathcal{A}_{A_y}(A_z)\mathbf{e}_{A_y}$ the pseudo vector potential of the pseudo-magnetic field, the Lagrangian of the pseudo-particle, which reads $L = \frac{1}{2}\left[(dA_y/dx)^2 + (dA_z/dx)^2\right] + \mathcal{A}_{A_y}dA_y/dx$, is then independent of A_y and consequently the pseudo-canonical momentum conjugate with A_y, i.e. $\mathcal{P}_{A_y} = B_z - \int\Lambda(A_z)dA_z$, is a constant of motion. A relevant example of a corresponding FF field that belongs to this class is the Harris sheet (7). In this case $\Lambda(A_z) = -\exp(A_z)$ and the conserved pseudo-canonical momentum, in the FF context, can be expressed as $\mathcal{P}_{A_y} = B_z - \Lambda(A_z)$. An example of a trajectory of a pseudo-particle corresponding to the Harris sheet is shown in Fig. 6. A further relevant example in this class is provided by the pseudo-particles moving close to a neutral line of the pseudo-magnetic field. This case has been investigated by Parker [32]. It corresponds to the choice $\Lambda(A_z) = -A_z$ and leads

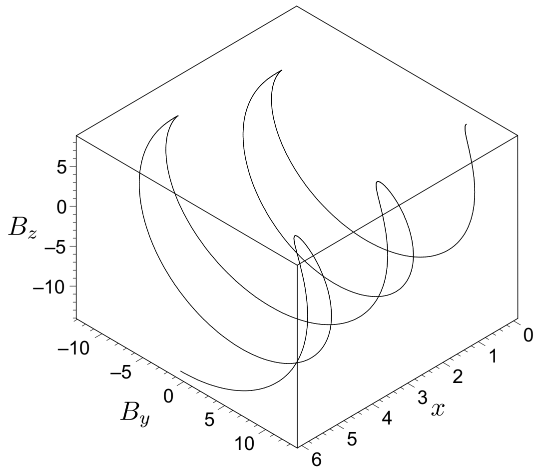

Fig. 7 Graph of a solution for **B** for a FF field corresponding to a pseudo-particle moving close to a neutral line. The solution refers to the initial conditions $A_z(0) = 0$, $A_y(0) = 0$, $B_z(0) = 7.1$, $B_y(0) = 8.1$.

to meandering orbits of the pseudo-particle and, correspondingly, of the vector potential of a FF field. An example of solution for the FF field corresponding to this pseudo-configuration is shown in Fig. 7.

4.3 $2D$ pseudo-configurations

These refer to Λ depending explicitly on A_y and A_z. The corresponding system is, in general, non-integrable and this leads to stochastic trajectories of the pseudo-particles and of the corresponding FF field orientation in the B_yB_z plane.

5. Conclusions

A number of new results on nonlinear FF fields have been shown. Several families of physically relevant FF fields described by Liouville equation have been presented. Classical solutions such as the Gold-Hoyle field, the Harris sheet and "cat's eye" solution belong to these families. After recalling the relation between the Lie symmetries of Liouville equation and its general solution, Lie symmetries for the scalar equation describing FF fields in helical symmetry have been derived. Aside from the limit $kr \ll 1$, where a non trivial symmetry of the equation has been found, the case with generic $g(\chi)$ only possesses a translational symmetry, unless $k = 0$, which brings us back to the case treated in Sec. 1.

Finally a formal correspondence between the equations governing $1\frac{1}{2}D$ FF fields in Cartesian coordinates and charged particles in a magnetic field has been shown. Within this analogy linear FF fields correspond to pseudo-particles in a uniform pseudo-magnetic field, whereas the Harris sheet is analogous to a pseudo-particle subject to a field varying exponentially with A_z. Conversely, the Parker problem of a charged particle near a neutral line maps into a nonlinear FF field with B_z depending quadratically on A_z. This analogy can be applied in further cases in order to transfer knowledge from the theory of motion of a charged particle (for instance what is known about drift approximation, adiabatic invariants,...) into the FF field context.

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