J. Plasma Fusion Res. SERIES, Vol. 8 (2009)

Vlasov simulation of finite amplitude magnetohydrodynamic waves in the solar wind : Development of Vlasov-Hall-MHD code

Takashi KUMASHIRO, Tohru HADA, Yasuhiro NARIYUKI[1)] and Takayuki UMEDA[2)]

E.S.S.T., Kyushu University, 6-1, Kasuga-koen, Kasuga City, Fukukoka 816-8580, Japan

[1)] *Kochi National College of Technology, Kochi 783-8508, Japan*

[2)] *Solar-Terrestrial Environment Laboratory, Nagoya University, Nagoya, Aichi 464-8601, Japan*

(Received: 6 September 2008 / Accepted: 16 January 2009)

Vlasov simulation is a method to solve time evolution of a plasma by directly time advancing the distribution function in the position-velocity phase space. Vlasov simulation is free from thermal (numerical) noise and thus is advantageous in analyzing fine details of nonlinear plasma phenomena. With this background in mind, we have developed a new Vlasov simulation code (1-d in the real space, 3-d in the velocity space), in order to study basic properties of nonlinear evolution of large amplitude magnetohydrodynamic (MHD) waves in the solar wind. In contrast to traditional Vlasov simulations in which electron waves are of major concern, our simulation code focuses on solving plasma behavior around the ion scales, assuming a massless electron fluid. Since we mainly deal with low frequency MHD waves propagating quasi-parallel to the background magnetic field, cyclotron coupling can be assumed to be weak for parameters typical to the solar wind. Thus, in our code, the Vlasov equation is solved only along the longitudinal direction, whereas the Hall-MHD equations are solved for the transverse directions. Propagation properties of Alfvén and ion acoustic waves in the simulation agree with analytically obtained dispersion relations. Some results on the parametric decay instability of Alfvén waves are also reported.

Keywords: Vlasov simulation, Hall-MHD, Alfvén waves, Parametric instability, solar wind

1. Introduction

Vlasov simulation is a numerically stable method to solve time evolution of a plasma by directly time advancing the distribution function in the position-velocity phase space [1]. While particle simulations, such as the PIC (particle-in-cell) simulations, are widely used for studies of kinetic properties of a plasma, these methods cannot avoid rather large statistical fluctuations (numerical noise) arising from the use of finite number of particles. It is particularly important to retain the noise level low for treating long time evolution of nonlinear systems (such as the parametric instabilities of Alfvén waves) since the presence of (large) noise can qualitatively alter the eventual state of system evolution. To keep numerical noise low, a large number of particles are needed. Unlike conventional PIC simulations, the Vlasov simulation is free from thermal (numerical) noise, and is thus advantageous in analyzing fine details of nonlinear plasma phenomena. On the other hand, Vlasov simulations require huge computational resource, especially for multi-dimensional models. Recent advances both in computing power and computational schemes, however, make it possible to execute relatively large scale calculations. It is expected that the Vlasov simulations will be used for increasingly wider range of applications in the near future.

With this background in mind, we have developed a new Vlasov simulation code (1-d in configuration space, 3-d in the velocity space), in order to study basic properties of nonlinear evolution of magnetohydrodynamic (MHD) waves, in particular, quasi-parallel (to the background magnetic field) propagating Alfvén waves in the solar wind. Quasi-parallel propagating Alfvén waves are thought to be ubiquitous in many astrophysical situations, for instance in the solar wind, solar corona, interstellar matter, magnetosphere, among others, and are thought to play important roles in the physical processes in these regions. [2]. Because they are typically robust for linear ion-cyclotron damping due to their small wave frequencies and for linear Landau damping due to their small propagation angle relative to the background magnetic field, Alfvén waves are thought to propagate very long distance [3]. Thus, the dissipation process of these Alfvén waves is important from a viewpoint of the momentum and energy transport in the solar wind. Among various possible scenarios for the Alfvén waves to dissipate, the parametric instabilities are regarded as one of the most robust and important processes responsible for the dissipation of the solar wind Alfvén waves.

While propagation of MHD waves can be under-

author's e-mail: kuma-taka@esst.kyushu-u.ac.jp

stood in a fluid model, inclusion of ion kinetic effects is essential in correctly describing the properties of the solar wind Alfvén waves since the ion pressure is about the same order as the magnetic pressure [4]. To be more precise, the ion kinetic effects have to be taken into account unless β_i is very small (for Alfvén waves) or β_e/β_i is very large (for ion acoustic waves). In the above, β_i and β_e are the ratio of the plasma pressure and the magnetic pressure for ions and electrons. Also, electron kinetic effects can be important in case of nearly perpendicular propagating Alfvén waves (kinetic Alfvén waves), although we do not discuss them in the present paper.

An ultimate goal of the Alfvén wave simulation may be to perform an extremely large scale simulation including the entire 3+3=6 dimensions in the configuration and the velocity phase space. However, computations in such a system require huge memory size and computation time, and are not realistic at present. Thus we focus on parallel propagating solar wind Alfvén waves and reduce required memory size and computational time by assuming the following three points. (1) As mentioned above, dynamics of the quasi-parallel propagating MHD waves is mainly controlled by ions, and so we treat electrons as a massless charge neutralizing fluid, in a way similar to the well-known hybrid code [5, 6], and a Vlasov code recently proposed by Valentini *et. al.*[7]. In our simulation, the Hall-MHD equation set is used to include the dispersiveness of the MHD waves for transverse directions (see point (3) below). (2) We assume that wave propagating directions do not change during the time scale of our interest, and limit ourselves to the 1-dimensional problem (1-d in the real, 3-d in the velocity space). Underlying here is the fact that parallel Alfvén waves stay dominant in the course of parametric instabilities of Alfvén waves we treat in the present paper [8]. Many kinds of important and outstanding problems could be considered even in the restricted phase space assumed in the 1-d system. (3) Since we mainly deal with low frequency Alfvén waves propagating parallel to the background magnetic field, cyclotron coupling can be assumed to be weak for typical parameters of the solar wind [3]. In our simulation code (Vlasov-Hall-MHD code), the Vlasov equation is solved only along the longitudinal direction whereas the Hall-MHD equations are solved for the transverse directions.

By employing these assumptions, ion distribution function can be represented only in the 1+1=2 dimensional phase space. Transverse ion velocities and electron dynamics are described in terms of fluid (averaged) variables. In order that the above assumptions remain valid throughout the simulation runs, conditions below have to be satisfied. First, parameters for plasma and waves are within the range in which energy exchange by cyclotron resonance can be negligible. Further, magnetic field direction at any point has to remain roughly parallel to the main axis (the x direction). For this sake, amplitude of the Alfvén waves should not be too large.

In this sense, the concept of the present Vlasov-Hall-MHD model is similar to Landau fluid model, in which leading order finite Larmor radius (FLR) corrections to the pressure are retained while contributions of non-gyrotropic heat flux components are neglected. It is reported that numerical integration of the Landau fluid model can accurately describe Alfvén wave instabilities [9, 10]. In a similar way, we expect that the present Vlasov-Hall-MHD model can correctly describe longitudinal characteristics of the Alfvén wave evolution.

The paper is organized as follows: in the next section, we describe in some detail the Vlasov-Hall-MHD model. In Section 3, we present our simulation results on dispersion relation of the ion acoustic and the Alfvén waves. In Section 4, we discuss Alfvén wave parametric instabilities, and mention briefly an interesting ion acceleration took place during the course of the Alfvén wave decay instability. Section 5 is a summary and conclusions.

2. Vlasov-Hall-MHD code

In this section we briefly describe our Vlasov-Hall-MHD model. Let us consider a phase space which is 1-d in the configuration space (x direction) and 3-d in the velocity space. The background magnetic field (=constant) is along the x direction, and we assume that local orientations of the magnetic field remain 'nearly' parallel to the x-axis everywhere as well, *i.e.,* transverse magnetic field due to waves/fluctuations is small. Then, the ion distribution function $F(x, t, \mathbf{v})$ may be separated into the longitudinal and the transverse directions, *i.e.,* $F(x, t, \mathbf{v}) = f(t, x, v_x)g(t, x, v_y, v_z)$. Note that the longitudinal and the x directions are interchangeably used here. Longitudinal and transverse distribution functions are normalized as follows,

$$\int f(t, x, v_x)dv_x = \rho(t, x) \tag{1}$$

and

$$\int g(t, x, v_y, v_z)dv_y dv_z = 1 \tag{2}$$

where $\rho(t, x)$ is the plasma density, and the integration is for the entire velocity space (for each variable). Hence,

$$f(t, x, v_x) = \int F(t, x, \mathbf{v})dv_y dv_z \tag{3}$$

From now on, we normalize all the physical variables using

Alfvén velocity, background magnetic field strength, and the ion cyclotron angular frequency (with ion sense of rotation being the negative frequency), all defined at a certain reference point.

Integration of the Vlasov equation over v_y and v_z yields,

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + (e_x + u_y b_z - u_z b_y)\frac{\partial f}{\partial v_x} = 0 \quad (4)$$

where $\mathbf{e} = (e_x, e_y, e_z)$ and $\mathbf{b} = (b_x, b_y, b_z)$ are the normalized electric and magnetic field, respectively. The ion bulk velocity, $\mathbf{u} = (u_x, u_y, u_z)$, is given by

$$\mathbf{u} = \frac{1}{\rho}\int \mathbf{v}F(t, x, \mathbf{v})dv_x dv_y dv_z \quad (5)$$

The perpendicular velocity evolves according to,

$$\frac{\partial u_\perp}{\partial t} = -u_x \frac{\partial u_\perp}{\partial x} + \frac{1}{\rho}\frac{\partial b_\perp}{\partial x} \quad (6)$$

where for brevity we have introduced complex perpendicular magnetic field, $b_\perp = b_y + ib_z$, and complex perpendicular velocity, $u_\perp = u_y + iu_z$. Then, Maxwell equations and the electron equation of motion lead to

$$e_x = -u_y b_z + u_z b_y - \frac{1}{\rho}\frac{\partial}{\partial x}(\frac{|b_\perp|^2}{2} + p_e) \quad (7)$$

$$\frac{\partial b_\perp}{\partial t} = -\frac{\partial}{\partial x}(u_x b_\perp - u_\perp + \frac{i}{\rho}\frac{\partial b_\perp}{\partial x}) \quad (8)$$

In the above, electrons are treated as a massless, isotropic fluid, quasi-charge neutrality is assumed, and the displacement current in neglected. Further assuming that the electron fluid is isothermal, $p_e = \rho T_e$, from Eqs.(4) and (7) we have

$$\frac{\partial f}{\partial t} = -v_x \frac{\partial f}{\partial x} + \frac{1}{\rho}\frac{\partial}{\partial x}(\frac{|b_\perp|^2}{2} + T_e\rho)\frac{\partial f}{\partial v_x} \quad (9)$$

Then, Eqs.(1), (5), (6), (8), (9) constitute a closed set (b_x is constant), which is the basic equation set of our model.

Next we briefly describe computational technique used in the Vlasov-Hall-MHD code. The method we employ for time advancement of the Vlasov equation is the PIC scheme (Positive Interpolation for hyperbolic Conservation laws) [12, 13], which is a recently developed high order interpolation scheme proposed for Vlasov simulations. The PIC scheme is shown to be particularly accurate and efficient for highly nonlinear problems [13], and is suited for the present study since long time evolution of Alfvén wave parametric instabilities and associated ion dynamics are strongly nonlinear and delicate problems (small numerical errors can lead to qualitatively different results). For the present study, we used periodic boundary conditions for x and free boundary conditions for v_x.

3. Dispersion relation of ion acoustic waves and Alfvén waves

To ascertain whether the present Vlasov-Hall-MHD code runs correctly, we first made thermal runs and evaluated the dispersion relations, and compare them with analytical dispersion relations. In case of parallel propagation, MHD waves decouples into longitudinal (ion acoustic) and transverse (Alfvén) waves. Strictly speaking, non-propagating entropy waves should also be included, but we do not consider them here.

Assuming Maxwellian for ions,

$$f_0(v_x) = \frac{1}{\sqrt{\pi}a_i}\exp\left(-\frac{{v_x}^2}{{a_i}^2}\right) \quad (10)$$

where $a_i^2 = 2T_i$ is ion thermal velocity squared, dispersion relation of the ion acoustic waves is given [14].

$$Z(\xi_i)' = \frac{2T_i}{T_e}, \quad (11)$$

where $\xi_i = \omega/ka_i$, ω is the angular wave frequency, k is the wave number, and $Z(\xi)$ is the plasma dispersion function. Assuming $\omega/k < a_i$, expansion of Eq.(11) yields

$$\omega_r = \pm v_s k = \pm(T_e + 3T_i)^{1/2}k \quad (12)$$

$$\frac{\omega_i}{|\omega_r|} = -\left(\frac{\pi}{8}\right)^{\frac{1}{2}}\left(\frac{T_e}{T_i}\right)^{\frac{3}{2}}\exp\left(-\frac{T_e}{2T_i} - \frac{3}{2}\right) \quad (13)$$

where $\omega = \omega_r + i\omega_i$. Unless ion temperature is much less than that of electrons, ion acoustic waves suffer from Landau damping Eq.(13).

Next we consider Alfvén waves in the Vlasov-Hall-MHD system. If we consider parallel propagating, circularly polarized waves, longitudinal parameters including the ion distribution function remain constant, and so we only need to consider the transverse fluid variables. Then the linear dispersion relation of the Alfvén waves is derived from Eqs.(6), (8).

$$\omega^2 = k^2(1 + \omega) \quad (14)$$

Since the parallel circular polarized Alfvén waves completely decouple from longitudinal ion dynamics, the unmodulated, parallel propagating Alfvén waves do not suffer from the Landau damping. Since in our normalization the ion cyclotron angular frequency is -1, the positive and negative ω corresponds to whistler-mode waves (right-hand polarized) and ion-cyclotron wave (left-hand polarized), respectively.

For the thermal run, the initial ion distribution function f_0 was given as a superposition of Eq.(10) and small perturbations. The results of thermal run are shown in Fig.1. Used parameters are $T_e = 0.16$, $v_s = 0.4$, $\beta_e = 0.32$, $a_i = 0.1$. Figures show (Top) time evolution of the transverse magnetic field and (Bottom) time evolution of the density. Small amplitude Alfvén waves and ion acoustic waves are seen to

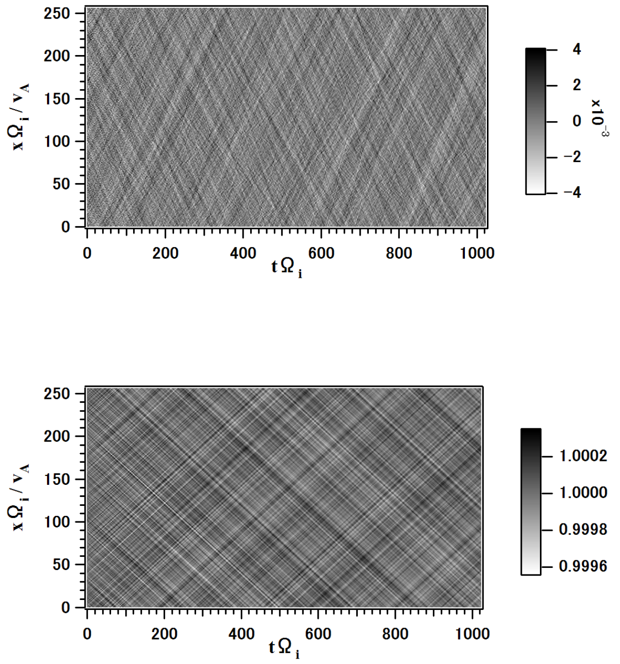

Fig. 1 Results of the thermal run. (Top) Time evolution of one transverse component of the magnetic field, b_y, which represents presence and propagation of Alfvén waves. (Bottom) Time evolution of the density, ρ, which basically represents the presence and propagation of the ion acoustic waves.

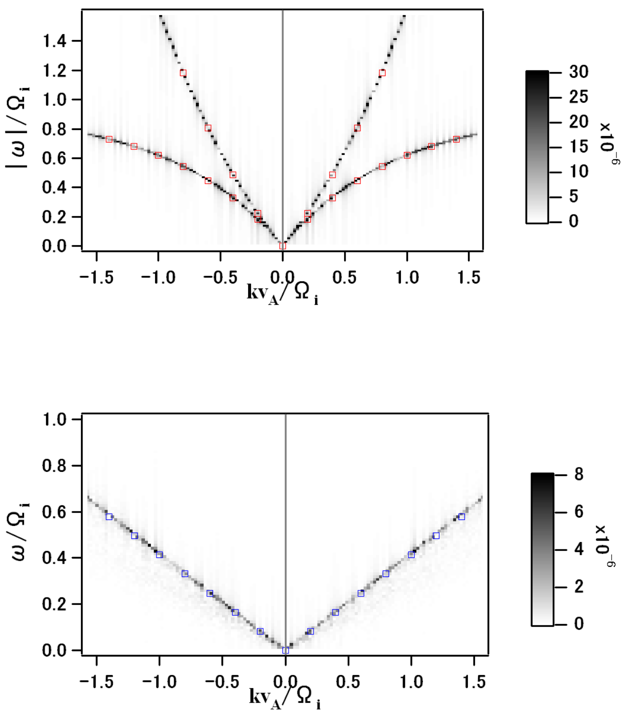

Fig. 2 Dispersion relations obtained from the thermal run shown in Fig.1. (Top) Dispersion relation obtained for the Alfvén waves. (Bottom) The same for the ion acoustic waves. In both figures, analytical dispersion relations Eq.(12) and Eq.(14)) are superposed as squares.

propagate with constant velocities and in both positive and negative directions. By Fourier transforming these variables above, we obtain the dispersion relation as shown in Fig.2. The results are in good agreement with the analytical expressions, Eqs.(12), (14).

4. Parametric instability of Alfvén waves

Alfvén waves play essential roles in many space plasma phenomena. In particular, they are believed to contribute in heating and acceleration of plasmas via parametric instabilities[15, 16, 17]. The parametric instabilities are categorized into decay type ($k_0 < k$) and self-modulational type ($k_0 > k$) instabilities according to the wave number of excited ion acoustic daughter waves, where k_0 is the wave number of the parent Alfvén wave and k is that of the ion acoustic wave. In this paper we chose the set of simulation parameters in such a way that the decay type is the dominant instability. As initial conditions, we give monochromatic, right-hand polarized, parallel propagating Alfvén waves in the system,

$$b_\perp = b_0 \exp\left[i\left(\omega_0 t - k_0 x\right)\right] \tag{15}$$

$$u_\perp = u_0 \exp\left[i\left(\omega_0 t - k_0 x\right)\right] \tag{16}$$

where $\omega_0^2 = k_0^2(1+\omega_0)$, $u_0 = -b_0/v_{p0}$, and $v_{p0} = \omega_0/k_0$ (Walen relation). Also we let initially $\rho = \rho_0 = 1$, $u_x = 0$, and $f = f_0$ as given by Eq.(10). The parent wave propagates toward the positive x direction.

We first check that the present Vlasov-Hall-MHD code correctly solves nonlinear evolution of Alfvén waves by running it as well as a Hall-MHD code (fluid dynamics both in the longitudinal and transverse directions) and a hybrid code (fluid electrons and particle-in-cell ions) using the same set of physical parameters (typical for the decay instability), and by making comparison the decay instability growth rates. Physical parameters used are: electron beta $\beta_e = 0.72$, ion beta $\beta_i = 0.01$ (almost cold), (*i.e.*, ion thermal velocity $a_i = 0.1$, electron temperature $T_e = 0.36$, ion temperature $T_i = 0.005$), normalized parent Alfvén wave amplitude $b_0 = 0.5$, wave number of the parent wave $k_0 = 0.417$, (which leads to) the parent wave frequency $\omega_0 = 0.513$. Other important parameters are: number of grids is 1024 in the real space and 1000 in the velocity space, system size = 256 (number of parent waves in the system $m = L/(2\pi/k_0) = 17$), velocity space grid size $\Delta v = 0.008$, time step $\Delta t = 0.01$, and the maximum time duration computed is $t_{max} = 250$.

From the Vlasov-Hall-MHD simulation run we first confirmed that the ion acoustic waves (k) and the backward propagating Alfvén waves (Stokes waves, $k - k_0 < 0$) were generated at respective wave number regimes expected from the theory. Higher harmonic waves of the Alfvén waves were also observed.

In addition to the Vlasov-Hall-MHD simulation, we have also performed Hall-MHD and hybrid simulations. Table 1 compares the maximum growth rates evaluated from the three different types of simulations as well as the (fluid) theory, using the same set of physical parameters. The growth rates for all the simulation runs are approximately equal. We conclude from the present test simulation that our new Vlasov-Hall-MHD code can provide quantitatively reliable results for our study on Alfvén wave evolution. The computational load of the Vlasov-Hall-MHD code is almost the same as the Hall-MHD code and much less than the hybrid code or full-Vlasov code. Despite this, time evolution of the ion distribution function is clearly obtained, which is one of the greatest advantages of the Vlasov-Hall-MHD code.

Type of code	Maximum growth rate (γ)
Hall-MHD	0.0690
hybrid	0.0646
Vlasov-Hall-MHD	0.0632
Analytical solution	0.0682

Table 1 Comparison of the decay instability maximum growth rates evaluated by various simulation runs and the analytical theory of Vlasov-Hall-MHD system.

Next we briefly discuss evolution of the ion distribution function due to the decay instability of right-hand polarized Alfvén waves (Fig.3). Parameters used are the same as before, except that $\beta_e = 0.32$, $b_0 = 0.4$, $k_0 = 0.196$ ($\omega_0 = 0.216$), and $m = L/(2\pi/k_0) = 8$. The upper part of Fig.3 (a) shows a snapshot of the ion distribution at an early stage of the decay instability ($t = 270$). Modulation of the ion distribution is clearly seen due to the longitudinal electric field (the bottom part of the figure) of the ion acoustic waves. In particular, slashing, steepening, and trapping of ions are evident in association with the large amplitude longitudinal electric field. Upper part of Fig.3 (b) shows a snapshot of ion distribution at the later phase of the evolution ($t = 770$), where the first decay instability (i.e., decay of the initial parent Alfvén wave: successive decay of daughter waves may be in progress but not seen evidently) is saturated. Around $x \sim 150$, some ions are strongly accelerated toward negative v_x direction. This acceleration is associated with an enhanced magnetic field modulation generated by the beating between parent and the backward propagating daughter Alfvén waves (the lower part of Fig.3 shows the magnetic field). The same type of ion acceleration can be found also for the decay of the left-hand polarized Alfvén waves. Details of the ion acceleration mechanism, as well as its relationship to the ion acceleration reported in hybrid simulations [11], will be discussed in a separate paper.

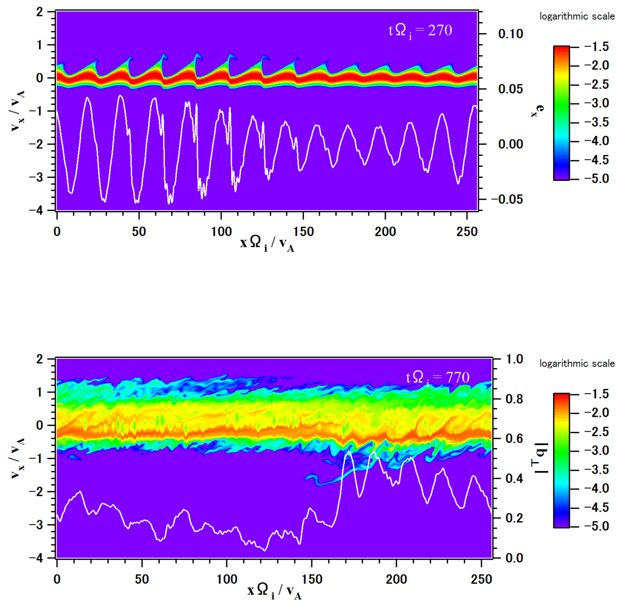

Fig. 3 (a) The ion distribution function (the upper part) and the longitudinal electric field (the lower part) plotted at an early stage of the simulation, $t = 270$. Ion acoustic waves generated by the decay instability strongly modulate the ion distribution function. (b) The ion distribution function (the upper part) and the magnetic field magnitude (the lower part) plotted at a later stage, $t = 770$. Some ions are strongly accelerated ($x \sim 150$), in association of enhanced magnetic field perturbation.

5. Summary

In this paper we have introduced our newly developed Vlasov simulation code (Vlasov-Hall-MHD code), in which the Vlasov equation is solved only for the longitudinal direction and the Hall-MHD set of equations are solved for the transverse directions. The electrons are treated as a massless fluid. The code is particularly suited for solving nonlinear long time evolution of quasi-parallel MHD waves, where transverse kinetic effects (such as cyclotron damping) can be neglected. Below we summarize results of some of our initial simulation runs.

(1) By performing thermal runs, we confirmed that the ion acoustic and the Alfvén waves, which satisfy the linear dispersion relation, are present in the simulation system,

(2) By giving circularly polarized, monochromatic Alfvén waves plus a very small random fluctuations as initial conditions, we simulated the evolution of Alfvén wave decay instability. All the daughter waves are generated, as well as the higher harmonics.

(3) Comparison of growth rates was made, by

running the present Vlasov-Hall-MHD code, the Hall-MHD code, and the hybrid code, using the same set of physical parameters. The present code is confirmed to yield reliable result.

(4) It is demonstrated that fine details of the ion distribution function can be captured by the present simulation code. In particular, an interesting ion acceleration was found to take place in association with the decay instability, for both right- and left-hand polarized parent waves.

Details of the ion acceleration, modulational instabilities, and conprehensive arguments on the Alfvén wave parametric instabilities (with a particular emphasis on longitudinal ion kinetic effects) will be discussed in a forthcoming paper.

Acknowledgment

This work was in part supported by Grant-in-Aid for Young Scientists (Start-up) No.19840024 from JSPS.

[1] C. Z. Cheng and G. Knorr, The integration of the Vlasov equation in configuration space, J. Comp. Phys. (1976), 22, 330.

[2] R. Bruno and V. Carbone, The solar wind as a turbulence laboratory, Living Rev. Solar Phys. (2005), 2. 4.

[3] S. P. Gary and J. E. Borovsky, Alfvén-cyclotron fluctuations: Linear Vlasov Theory, J. Geophys. Res. (2004), 109, doi:10.1029.

[4] Y. Nariyuki and T. Hada, Consequences of finite ion temperature effects on parametric instabilities of circularly polarized Alfvén waves, J. Geophys. Res. (2007), VOL. 112, A10107

[5] D. Winske, Hybrid simulation codes with application to shocks and upstream waves, Space Sci. Rev. (1985), 42 (1-2), doi. 10.107/BF00218223.

[6] T. Terasawa, M. Hoshino, J.-I. Sakai and T. Hada, Decay instability of finite-amplitude circulary polarized Alfvén waves: A numerical simulation of stimulated Brillouin Scattering, J. Geophys. Res. (1986), 91, 4171 -4187.

[7] F. Valentini, P. Travnicek, F. Califano, P. Hellinger, A. Mangeney, A hybrid-Vlasov model based on the current advance method for the simulation of collisionless magnetized plasma, J. Comp. Phys. (2007), 225, 753-770.

[8] Y. Nariyuki, S. Matsukiyo and T. Hada, Parametric instabilities of large-amplitude parallel propagating Alfvén waves : 2D PIC simulation New journal of Phys. (2008), doi. 10.1088/1367-2630/10/8/083004

[9] G. Bugnon, T.Passot, and P. L. Sulem, Landau-fluid simulations of Alfvén-wave instability in a warm collisionless plasma, Nonl. Proc. Geophys. (2004), 11, 609-618.

[10] T.Passot, and P. L. Sulem, A landau-fluid model for dispersive magnetohydrodynamics, Phys. Plasma, (2004), 11, 5173-5189.

[11] J. A. Araneda, E. Marsch, A. F. Vinas, Proton core heating and beam formation via parametrically unstable Alfvén-cyclotron waves, Phys. Rev. Lett. (2008), 100, 125003.

[12] T. Umeda, M. Ahsour-Abdalla and D. Schriver, Comparison of numerical interpolation schemes for one-dimensional electrostatic Vlasov code, J. Plasma Phys. (2006), 72, 1057-1060.

[13] T. Umeda A conservative and non-oscillatory scheme for Vlasov code simulations Earth Planets Space, (2008), 60(7), 773-779.

[14] F. F. Chen Introduction to plasma physics and controlled fusion, vol.1, "Plasma Physics", (1984), Plenum Press, New York

[15] A. A. Galeev, V. N. Oraevskii, The instability of Alfvén waves, Sov. Phys. Dokl. (1963), 7, 988-1003.

[16] M. L. Goldstein, An instability of finite amplitude circularly polarized Alfvén waves, Astrophys. J. (1978), 219 (2), 700.

[17] Y. Nariyuki and T. Hada, Parametric instabilities of parallel propagating incoherent Alfvén waves in a finite ion beta plasma, Phys. Plasma, (2007) 14, 122110.