Gradient-Length Analysis of the Magnetized Plasma-Wall Transition (MPWT)

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For analytic simplicity, the magnetized plasma-wall transition (MPWT) layer occurring, e.g., near tokamak divertors, is usually split into three sublayers: the Debye sheath (DS), the magnetic presheath (MPS) and the collisional presheath (CPS), with characteristic (indicated by superscript "ch") length scales λ_D^{ch} (electron Debye length), ρ_i^{ch} (ion gyro-radius) and λ_c^{ch} (smallest relevant ion collision length), respectively. The values of these characteristic lengths are taken at certain characteristic points of the respective sublayers, whereas the gradient lengths vary with position. We introduce a new method for systematically investigating the behavior of gradient-length scales in the MPWT. The gradient lengths are used to replace in the fluid equations all gradients, which transforms these equations into an algebraic system. Here, as a first example of application, this method is applied to the MPWT in the asymptotic three-scale (A3S) limit $\lambda_D^{ch} \ll \rho_i^{ch} \ll \lambda_c^{ch}$.

Keywords: Plasma-wall transition, Debye sheath, magnetic presheath, collisional presheath, plasma-wall interaction, tokamak plasma.

1. Introduction

In the presence of a magnetic field oblique to the wall, the plasma-wall transition (PWT) layer can be divided into three regions, namely: Debye sheath (DS), magnetic presheath (MPS) and collisional presheath (CPS) [1]. In the classical PWT problem without magnetic field, monotonicity of the electric potential requires fulfillment of the Bohm condition at the interface between the CPS (with the characteristic collisional length $\sim \lambda_c^{ch}$, chosen appropriately) and the DS ($\sim \lambda_D^{ch}$). Chodura was the first to investigate the MPS in an oblique magnetic field without any collisional effects [2]. The MPS was found to scale with the characteristic ion gyro-radius ρ_i^{ch} For the limiting ordering of the scale variables, i.e., $\lambda_D^{ch} << \rho_i^{ch} << \lambda_c^{ch}$, the problem contains three distinguished scales and, in this "asymptotic three-scale (A3S) limit", i.e. for $\varepsilon_{Dm} \equiv \lambda_D^{ch} / \rho_i^{ch} \to 0$ and $\varepsilon_{mc} \equiv \rho_i^{ch} / \lambda_c^{ch} \to 0$, the DS can be characterized as collisionless and non-neutral, the MPS as collisionless and quasi-neutral ($n_i = n_e$) [2] and the CPS as collisional and quasi-neutral.

The DS and the MPS regions are separated by the sheath edge. From the MPS side it is characterized by a field singularity, and from the DS side by the marginal form of the Bohm condition, $u_z = c_s = \left[k\left(T_e + \gamma T_i\right)/m_i\right]^{1/2}$ (c_s is the ion-sound velocity, *k* is the Boltzmann constant, and γ(*z*) is the polytropic coefficient [3]). The MPS-CPS boundary surface is referred to as the "MPS entrance". Below we show that quite in analogy with the unmagnetized PWT, the MPS entrance can be defined as a point were the electric field has a singularity from the CPS side. The condition imposed from the MPS side is similar to the Bohm condition, but the ion velocity must be directed along the magnetic field line, $u_{\|} = c_s$. This is called as Bohm-Chodura condition. Hence the dominant effect of the MPS is to deflect the ion orbits, so that the velocity component u_z , can fulfill the Bohm condition at the DS entrance [1,4].

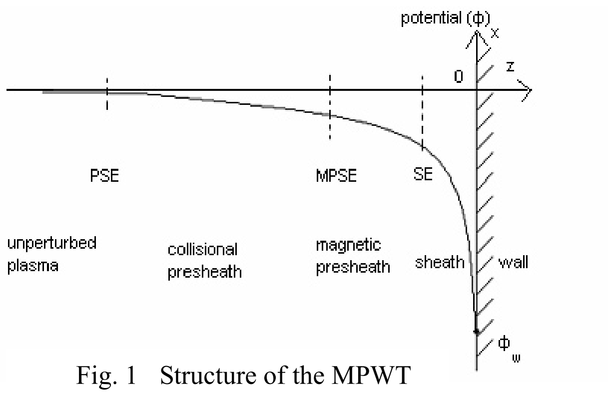

Fig. 1 Structure of the MPWT

2. Model and basic equations

The problem is 1D, with the *z* axis perpendicular to the wall surface placed at $z = 0$. The plasma occupies the region $z < 0$. The electric potential Φ decreases towards the wall monotonically. The uniform magnetic field is assumed to be lying in the $x - z$ plane, making a small angle α with the wall. The thermal motion of the ions is neglected, $T_i \to 0$

To describe the magnetized plasma-wall transition (MPWT), we choose the following set of basic equations and definitions.

Ion continuity equation:

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$$\frac{d}{dz}(n_i u_z) = \nu_i n_e - \kappa n_e n_i = \nu_{ir} n_i, \tag{2.1}$$

where ν_i is the "ionization frequency" (related to single electrons), κn_e is the "recombination frequency" (related to single ions), and

$$\nu_{ir} = \nu_i \, n_e / n_i - \kappa n_e \tag{2.2}$$

is the combined frequency at which single ions appear ($\nu_{ir} > 0$) or disappear ($\nu_{ir} < 0$) due to the combined effect of ionization and recombination.

Ion momentum equations:

$$u_z \frac{du_x}{dz} = \omega_z u_y - \bar{\nu} u_x, \tag{2.3}$$

$$u_z \frac{du_y}{dz} = \omega_x u_z - \omega_z u_x - \bar{\nu} u_y, \tag{2.4}$$

$$u_z \frac{du_z}{dz} = \frac{Ze}{m_i} E - \omega_x u_y - \frac{1}{m_i n_i} \frac{dp_i}{dz} - \bar{\nu} u_z, \tag{2.5}$$

with

$$\vec{\omega} = Ze \vec{B} / m_i \tag{2.6}$$

the ion-cyclotron-frequency vector and

$$\bar{\nu} = \nu_{cx} + \nu_{ir} \tag{2.7}$$

the combined charge-exchange, ionization and recombination frequency.

Electron density:

$$n_e = n_e(\phi), \tag{2.8}$$

with $n_e(\phi)$ a given function. The related "electron screening temperature" is defined as

$$T_{e*} = \frac{e n_e}{k (dn_e / d\phi)} \tag{2.9}$$

and, hence, may be considered a given function of ϕ as well.

Poisson's equation:

$$\frac{dE}{dz} = \frac{e}{\varepsilon_0}(Zn_i - n_e) \tag{2.10}$$

Ion polytropic law:

$$\frac{dp_i}{dz} = \gamma k T_i \frac{dn_i}{dz} \tag{2.11}$$

Ideal-gas law:

$$p_i = n_i k T_i \tag{2.12}$$

Electric field:

$$E = -d\phi / dz \tag{2.13}$$

In addition we introduce the electric charge density

$$\rho = e(Zn_i - n_e), \tag{2.14}$$

which, depending on convenience, may be used alternatively instead of n_e as a variable.

3. Gradient-length representation

The length scales on which the various quantities vary with z play a crucial role in determining the structure of the solutions. To formalize the related analysis, we rewrite gradients (i.e., derivatives with respect to z) in the form

$$\frac{dQ(z)}{dz} = \frac{Q^g(z)}{l_Q(z)}, \tag{3.1}$$

where $Q(z)$ may be any variable and $Q^g(z)$ is a typical value of Q suitable for defining the "local gradient length" l_Q at position z ,

$$l_Q(z) = \frac{Q^g(z)}{dQ(z)/dz}, \tag{3.2}$$

which in general may assume positive or negative values. Clearly, $|l_Q|$ may be interpreted as the distance over which the change in $|Q|$ is of the order of $|Q^g|$. Accordingly, we can write

$$\begin{aligned}
&\frac{du_x}{dz} = \frac{u_x^g}{l_{ux}}, \quad \frac{du_y}{dz} = \frac{u_y^g}{l_{uy}}, \quad \frac{du_z}{dz} = \frac{u_z^g}{l_{uz}}, \\
&\frac{dn_i}{dz} = \frac{n_i^g}{l_{ni}}, \quad \frac{dp_i}{dz} = \frac{p_i^g}{l_{pi}}, \quad \frac{dn_e}{dz} = \frac{n_e^g}{l_{ne}}, \\
&\frac{dE}{dz} = \frac{E^g}{l_E}, \quad \frac{d\phi}{dz} = \frac{\phi^g}{l_\phi}, \quad \frac{d\rho}{dz} = \frac{\rho^g}{l_\rho},
\end{aligned} \tag{3.3}$$

In addition we define the local ion gyroradius $\rho_i(z)$, the local electron Debye length $\lambda_{De}(z)$, as well as the local collisional lengths $l_i(z)$ ("ionization length"), $l_r(z)$ ("recombination length"), $l_{ir}(z)$ ("combined ionization and recombination length"), $l_{cx}(z)$ ("charge-exchange mean free path") and $\bar{l}(z)$ (combined CX-ionization and recombination length") by

$$\rho_i = \frac{u_z}{\omega} = \frac{u_z m_i}{ZeB} \qquad \lambda_{De} = \sqrt{\frac{\varepsilon_0 k T_{e*}}{n_e e^2}} \tag{3.4}$$

and

$$\begin{aligned}
&\frac{1}{l_i} = \frac{\nu_i \, n_e}{u_z \, n_i}, \qquad \frac{1}{l_r} = \frac{\kappa n_e}{u_z}, \\
&\frac{1}{l_{ir}} = \frac{\nu_{ir}}{u_z} = \frac{1}{l_i} - \frac{1}{l_r}, \qquad \frac{1}{l_{cx}} = \frac{\nu_{cx}}{u_z}, \\
&\frac{1}{\bar{l}} = \frac{\bar{\nu}}{u_z} = \frac{1}{l_{cx}} + \frac{1}{l_{ir}} = \frac{1}{l_{cx}} + \frac{1}{l_i} - \frac{1}{l_r},
\end{aligned} \tag{3.5}$$

respectively. These lengths may be considered to be the "natural" lengths existing in our problem. Let us also define the "dominant" (i.e., smallest) collision lengths as

$$l_c^{\min}(z) = \min\left\{|l_{ir}|, |\overline{l}|\right\}. \qquad (3.6)$$

With the definitions (3.3), Eqs. (2.1), (2.3) − (2.5) and (2.9) − (2.11) can be rewritten in the gradient-length form

$$\frac{1}{n_i}\frac{n_i^g}{l_{ni}} + \frac{1}{u_z}\frac{u_z^g}{l_{uz}} = \frac{1}{l_{ir}}, \qquad (3.7)$$

$$\frac{u_x^g}{l_{ux}} + \frac{u_x}{\overline{l}} = \frac{\sin\alpha}{\rho_i}u_y, \qquad (3.8)$$

$$\frac{u_y^g}{l_{uy}} + \frac{u_y}{\overline{l}} = \frac{\cos\alpha}{\rho_i}u_z - \frac{\sin\alpha}{\rho_i}u_x, \qquad (3.9)$$

$$\frac{1}{u_z}\frac{u_z^g}{l_{uz}} + \frac{1}{\overline{l}} = \frac{ZeE}{m_i u_z^2} - \frac{\cos\alpha}{\rho_i}\frac{u_y}{u_z} - \frac{1}{n_i m_i u_z^2}\frac{p_i^g}{l_{pi}}, \qquad (3.10)$$

$$\frac{T_{e*}}{n_e}\frac{n_e^g}{l_{ne}} = -\frac{e}{k}E, \qquad (3.11)$$

$$\frac{E^g}{l_E} = \frac{e}{\varepsilon_0}\left(Zn_i - n_e\right), \qquad (3.12)$$

and

$$\frac{1}{p_i}\frac{p_i^g}{l_{pi}} = \frac{\gamma}{n_i}\frac{n_i^g}{l_{ni}}. \qquad (3.13)$$

$$\frac{\phi^g}{l_E} = -E, \qquad (3.14)$$

respectively, and differentiating (2.14) we find

$$\frac{\rho^g}{l_\rho} = e\left(Z\frac{n_i^g}{l_{ni}} - \frac{n_e^g}{l_{ne}}\right) \qquad (3.15)$$

For later use we introduce the additional definitions

$$\frac{1}{\overline{l}_{ux}} = \frac{1}{u_x}\frac{u_x^g}{l_{ux}} + \frac{1}{\overline{l}}, \qquad \frac{1}{\overline{l}_{uy}} = \frac{1}{u_y}\frac{u_y^g}{l_{uy}} + \frac{1}{\overline{l}}, \qquad (3.16)$$

$$c_e^2 = \frac{ZkT_{e*}}{m_i}, \qquad c_i^2 = \frac{\gamma kT_i}{m_i},$$
$$c^2 = c_e^2 + c_i^2 = \frac{k}{m_i}\left(ZT_{e*} + \gamma T_i\right), \qquad (3.17)$$

and

$$\hat{\rho} = \frac{\rho}{Zen_i - \rho} = \frac{\rho}{en_e}, \qquad (3.18)$$

Equations (3.7) − (3.15) represent a system of 9 equations interrelating the 9 gradient lengths $l_{ux}(z)$, $l_{uy}(z)$, $l_{uz}(z)$, $l_{ne}(z)$, $l_{ni}(z)$, $l_{pi}(z)$, $l_E(z)$, $l_\phi(z)$, $l_\rho(z)$ and the 9 scalar functions $u_x(z)$, $u_y(z)$, $u_z(z)$, $n_e(z)$, $n_i(z)$, $p_i(z)$, $E(z)$, $\phi(z)$ and $\rho(z)$, which are the solution functions of the 9 original basic equations (2.1), (2.3) − (2.5), (2.9) − (2.11), (2.13) and (2.14). In particular, the gradient lengths can be expressed in terms of the solution functions as follows:

$$\frac{u_x^g}{l_{ux}} = \frac{\sin\alpha}{\rho_i}u_y - \frac{u_x}{\overline{l}}, \qquad (3.19)$$

$$\frac{u_y^g}{l_{uy}} = \frac{\cos\alpha}{\rho_i}u_z - \frac{\sin\alpha}{\rho_i}u_x - \frac{u_y}{\overline{l}}, \qquad (3.20)$$

$$u_z^g / l_{uz} = \frac{u_z u_z^2}{u_z^2 - c_i^2}\left(\frac{ZeE}{m_i u_z^2} - \frac{\cos\alpha}{\rho_i}\frac{u_y}{u_z} - \frac{1}{l_{ir}}\frac{c_i^2}{u_z^2} - \frac{1}{\overline{l}}\right), \qquad (3.21)$$

$$\frac{n_e^g}{l_{ne}} = -n_e\frac{eE}{kT_{e*}}, \qquad (3.22)$$

$$\frac{n_i^g}{l_{ni}} = \frac{n_i u_z^2}{u_z^2 - c_i^2}\left(\frac{\cos\alpha}{\rho_i}\frac{u_y}{u_z} - \frac{ZeE}{m_i u_z^2} + \frac{1}{l_{ir}} + \frac{1}{\overline{l}}\right), \qquad (3.23)$$

$$\frac{p_i^g}{l_{pi}} = \frac{\gamma p_i u_z^2}{u_z^2 - c_i^2}\left(\frac{\cos\alpha}{\rho_i}\frac{u_y}{u_z} - \frac{ZeE}{m_i u_z^2} + \frac{1}{l_{ir}} + \frac{1}{\overline{l}}\right), \qquad (3.24)$$

$$\frac{E^g}{l_E} = \frac{\rho}{\varepsilon_0}, \qquad (3.25)$$

$$\frac{\phi^g}{l_\phi} = -E, \qquad (3.26)$$

With the help of Eqs. (3.19), (3.20) and (3.22)–(3.26), Eq. (3.21) can be cast into the following form, which will be used below:

$$\begin{aligned}
&\frac{1}{u_z}\frac{u_z^g}{l_{uz}}\left(1 - \frac{c^2 + c_e^2\hat{\rho}}{u_z^2}\right) - \frac{c_e^2}{u_z^2}\left(\frac{1}{n_i}\frac{n_i^g}{l_{ni}} - \frac{1}{n_e}\frac{n_e^g}{l_{ne}}\right) \\
&\quad + \frac{c_e^2}{u_z^2}\frac{\lambda_{De}^2}{l_E l_{ni} l_{ne}}\frac{n_i^g}{n_i}\frac{n_e^g}{n_e} \\
&= -\frac{c^2 + c_e^2\hat{\rho}}{u_z^2}\frac{1}{l_{ir}} - \frac{1}{\overline{l}} - \frac{\overline{l}_{uy}\cos^2\alpha}{\rho_i^2 + \overline{l}_{ux}\overline{l}_{uy}\sin^2\alpha},
\end{aligned} \qquad (3.27)$$

4. Gradient-length analysis of the MPWT

4.1. *MPS entrance*

In the A3S limit, both the CPS and the MPS are quasineutral, so that in Eq. (3.27) we set $\hat{\rho} = 0$.

(i) From the CPS side, the relations between the gradient lengths are

$$\left|l_{ux}\right|,\ \left|l_{uy}\right|,\ \left|l_{uz}\right| = O\left(l_c^{ch}\right) \gg \rho_i. \tag{4.1}$$

From (3.27) we find

$$\frac{1}{u_z}\frac{u_z^g}{l_{uz}}\left(1-\frac{c^2}{u_z^2}\right)+\frac{\cos^2\alpha}{\sin^2\alpha}\frac{1}{u_x}\frac{u_x^g}{l_{ux}} = -\frac{1}{\sin^2\alpha}\frac{1}{l_{cx}}-\left(\frac{c^2}{u_z^2}+\frac{1}{\sin^2\alpha}\right)\frac{1}{l_{ir}}, \tag{4.2}$$

From (3.9) and (4.1) it follows that

$$l_{ux} = l_{uz}, \tag{4.3}$$

Then from (4.2) we find

$$\frac{1}{u_z}\frac{u_z^g}{l_{uz}}\left(\frac{c^2\sin^2\alpha}{u_z^2}-1\right)=\frac{1}{l_{cx}}+\left(\frac{c^2\sin^2\alpha}{u_z^2}+1\right)\frac{1}{l_{ir}}. \tag{4.4}$$

If the marginal form of the Bohm-Chodura condition,

$$u_z^2 = c^2\sin^2\alpha, \tag{4.5}$$

is fulfilled, we have $l_{ux} = 0$. This means that the derivative of the ion velocity (and the electric field) has a singularity. The point where the marginal Bohm-Chodura condition is fulfilled can be considered as the MPS entrance as seen from the CPS side.

(ii) Close to the MPS entrance, we have from the MPS side

$$\left|l_{ux}\right|,\ \left|l_{uy}\right|,\left|l_{uz}\right|,\rho_i \ll \left|l_{ir}\right|,\left|l_{cx}\right|, \tag{4.6}$$

The quasineutrality condition, $\hat{\rho}=0$, is again fulfilled. From Eq. (3.27) we have

$$\frac{1}{\tilde{l}_{uz}}\left(1-\frac{c^2}{u_z^2}\right)=-\frac{\tilde{l}_{uy}\cos^2\alpha}{\rho_i^2+\tilde{l}_{ux}\tilde{l}_{uy}\sin^2\alpha}, \tag{4.7}$$

where

$$\frac{1}{\tilde{l}_{uj}}=\frac{u_j^g}{u_j}\frac{1}{l_{uj}} \quad \text{and} \quad \frac{1}{\tilde{l}_\rho}=\frac{\rho^g}{\rho}\frac{1}{l_\rho}. \tag{4.8}$$

It can be shown that

$$\tilde{l}_{ux}=\tilde{l}_{uy}=\tilde{l}_{uz}, \tag{4.9}$$

with which straightforward calculations lead to the result

$$\frac{\rho_i^2}{\tilde{l}_{uz}}=\frac{\sin^2\alpha}{\cos^2\alpha}\left(1-\frac{c^2\sin^2\alpha}{u_z^2}\right). \tag{4.10}$$

Hence at

$$u_z^2 \to c^2\sin^2\alpha \quad \left(u_z^2 > c^2\sin^2\alpha\right) \tag{4.11}$$

we have

$$l_{uz}\to\infty, \tag{4.12}$$

which means that on the scale of the ion gyroradius the z velocity component tends monotonically towards a constant value. The same behavior can be shown to be true for the other physical quantities as well. The relation (4.11) is known as the non-marginal form of the Bohm-Chodura condition for the MPS entrance.

4.2. *DS entrance*

For the definition of the DS entrance, we can again use Eq. (3.27).

(i) For the analysis of the MPS-DS transition from the MPS side, we again assume, besides quasineutrality, the length relations (4.6). From (3.9), (3.10) and (3.27) we then obtain

$$\tilde{l}_{ux}=\frac{\rho_i u_x}{u_y\sin\alpha}, \tag{4.13}$$

$$\tilde{l}_{uy}=\frac{\rho_i u_y}{u_z\cos\alpha-u_x\sin\alpha}, \tag{4.14}$$

$$\frac{1}{\tilde{l}_{uz}}\left(1-\frac{c^2}{u_z^2}\right)=\frac{\tilde{l}_{uy}\cos^2\alpha}{\rho_i^2+\tilde{l}_{ux}\tilde{l}_{uy}\sin^2\alpha}, \tag{4.15}$$

Substitution of (4.13) and (4.14) into (4.15) yields

$$\frac{u_z^g}{u_z}\frac{1}{l_{uz}}\left(\frac{c^2}{u_z^2}-1\right)=\cos\alpha\frac{\omega u_y}{u_z^2}, \tag{4.16}$$

The condition

$$u_z^2=c^2 \tag{4.17}$$

is the Bohm condition in the marginal form, defining the singularity point where $l_{uz}=0$. This point is defined as the DS entrance in the A3S limit

(ii) From the DS side the quasineutrality is broken $(\rho\neq 0)$, and the gradient lengths satisfy the condition

$$\left|l_{ux}\right|,\ \left|l_{uy}\right|,\left|l_{uz}\right|,\left|l_\rho\right|,\left|l_E\right| \ll \rho_i,\left|l_{ir}\right|,\left|l_{cx}\right|, \tag{4.18}$$

From Eq. (3.27) we obtain

$$\frac{1}{\tilde{l}_{uz}}\left(1-\frac{c^2}{u_z^2}\right)=\hat{\rho}\frac{c_e^2}{u_z^2}\left(\frac{1}{\tilde{l}_{uz}}+\frac{1}{\tilde{l}_\rho}\right). \tag{4.19}$$

Using Eqs. (3.11), (3.12) and (3.18), we obtain from (4.19)

$$1-\frac{c^2}{u_z^2}=\frac{c_e^2}{u_z^2}\frac{\lambda_D^2}{l_E^2}\left(1+\frac{\tilde{l}_{uz}}{\tilde{l}_\rho}\right). \tag{4.20}$$

where the relation

$$-\tilde{l}_{ne}\simeq\tilde{l}_E \tag{4.21}$$

has been used, where $\tilde{l}_{ne}$ and $\tilde{l}_E$ are defined in analogy with (4.8). The lengths l_{uz} and l_ρ should have the same order, i.e. $l_{uz}\simeq l_\rho$. From Eq. (4.20) we obtain that at the point where

$$u_z^2 \to c^2 \quad \left(u_z^2 > c^2\right) \tag{4.22}$$

we have

$$l_E^2 \to \infty. \tag{4.23}$$

This means that on the scale of the Debye length the electric field on the DS side tends asymptotically towards a constant value. The same behavior can be shown to be true for the other physical quantities as well.

5. Conclusion

A new method for analyzing the magnetized plasma-wall transition (MPWT), based on local and characteristic gradient-length scales, has been introduced (Sec. 3). By means of this new method, the CPS-MPS and MPS-DS transitions have been investigated (Sec. 4) for a fairly general MPWT model (Sec. 2). It has been found for the first time that the MPS-CPS interface (MPS entrance) can be defined as a surface where the electric field from the CPS side has a singularity, quite in analogy with the CPS-DS transition of the unmagnetized PWT. Regarding the MPS-DS interface, previous results have been recovered. The present work adds a significant contribution to the analysis and understanding of the MPWT.

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