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Exact Periodic Solutions of the Stationary Hasegawa-Mima Equation

SPINEANU Florin, VLAD Madalina, ITOH Kimitaka[1] and ITOH Sanae -I.[2]

Association Euratom-MEC, Bucharest, Romania

[1]National Institute for Fusion Science, Toki 509-5292, Japan

[2]Research Institute for Applied Mechanics, Kyushu University, Kasuga 816-8580, Japan

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Abstract

The small scale vortical structures generated by the ion polarisation drift nonlinearity appear as an intermediate step in reorganisation of plasma flows (*e.g.* decays of zonal flows). We find that a lattice of vortices is an exact solution at stationarity of a model equivalent to the Hasegawa-Mima equation and provide explicit solutions.

Keywords:

ion drift wave, vortices, zonal flow, integrable structure

1. Introduction

Due to the strong nonlinear character the ion drift wave generates turbulent flows and also coherent structures [1-2]. The radially extended eddies of the Ion Temperature Gradient driven instability (ITG) are a major agent of transport of energy in tokamak plasmas. At space scales of the order of the ion Larmor radius (ρ_s), the polarisation drift nonlinearity induces condensation of large flows into small vortices, very similar to the superconducting media. There is also an intermediate space scale, of the order $\sqrt{\rho_s L_n}$ where the flows are dominated by the scalar nonlinearity [3]. In this range one finds monopolar vortices which represent a reminiscence of the anticyclonic monopoles at lower scales and are very robust without being solitons. At the same scale, exact solutions (with scalar nonlinearity) have precisely the geometry of the zonal flows.

The dynamics on the larger scales is structuraly unstable against the increase of presence of the polarisation drift nonlinearity. Higher level will induce decay of the flow into small vortices. However these vortices evolve by collisions and merging and at late time the flow can exhibit a regular pattern, a process similar to the Euler ideal fluid [4]. From this perspective, this nonlinearity acts to reorganize the flow and at early stages it traverses states of turbulence and random vortices. The target of this reorganisation is a state of the fluid which associates with the self-duality.

We are interested in the small scale vortical solutions of the Hasegawa-Mima (HM) equation [1] since the numerical simulations show that the exact zonal flow solution breaks down firstly into arrays of vortices. Such solutions are not known for HM equation. We prove their existence and provide the exact form.

2. Structural instability of the zonal flows pattern and decay into vortices

At intermediate scales the ion drift wave is dominated by the scalar (or KdV) nonlinearity, which leads to the Flierl-Petviashvili (FP) equation [5]

$$\Delta\varphi = \alpha\varphi - \beta\varphi^2 \tag{1}$$

where $\alpha = \frac{1}{\rho_s^2}\left(1 - \frac{v_*}{u}\right)$ and $\beta = \frac{T_e}{2u^2 e B_0^2 \rho_s^2}\frac{\partial}{\partial x}\left(\frac{1}{L_n}\right)$ (v_* and u are ion diamagnetic and respectively the plasma poloidal flow velocities, B_0 is the magnetic field, L_n is the density gradient length and T_e the electron temperature). We have identified an exact periodic solution [6]

$$\varphi_s(x, y) = \frac{\alpha}{2\beta} + s\wp\left(iay + ibx + \omega \,\middle|\, g_2 = \frac{3\alpha^2}{(s\beta)^2} \right) \tag{2}$$

with the condition $a^2 + b^2 = s\beta/6$ where $\wp$ is the doubly periodic elliptic Weierstrass function. The first parameter g_2 of $\wp$, is fixed by the physical parameters as shown, while s and the second parameter, g_3 are fixed by boundary conditions. When the FP equation is perturbed with the term of polarisation drift the flow evolves to an ensemble of vortices initially disposed in a regular array, Fig. 1 ([7] and references therein). It is not possible to connect this evolution to the Kelvin-Helmholtz instability in the absence of many elements of the Prandtl model (pressure asymmetry, Bernouli law) and the most suggestive explanation is connected with

Corresponding author's e-mail: spineanu@ifin.nipne.ro

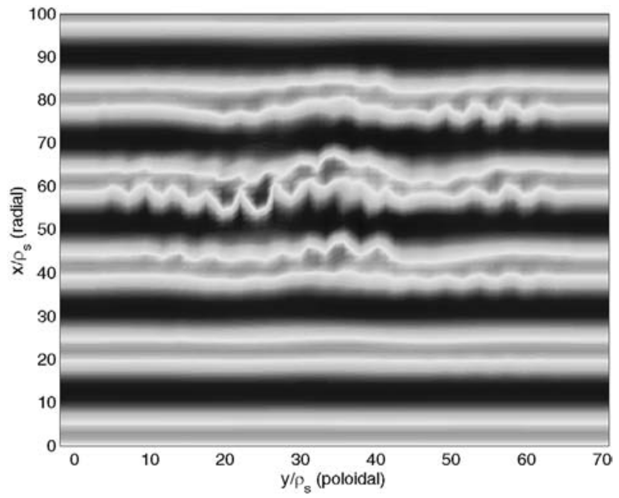

Fig. 1 A large wavelength inital perturbation evolves to a set of vortices.

the similar phenomenon in the superconducting media. This is also supported by the model we will propose below for the stationary HM.

3. The stationary states of the Hasegawa-Mima equation

The Euler equation has the solutions of the sinh-Poisson equation as stationary states. In the case of the HM eq., there is a similar model (developed in meteorology) where the dynamics is represented by the motion of point-like vortices in plane, interacting via a short range potential [2]. We formalize this model in a continuum version by a "matter" scalar field and a potential, asking that the density of "matter" tends at large distance to a condensate of vorticity, corresponding to the Larmor gyration. On this background, the solutions of the HM equation appear as excitations. The suitable model for this is the Abelian Higgs (AH) theory (developed for superconducting media) which generates the short range interaction (Kelvin functions with decay on ρ_s). This model clearly exhibits a property which is less transparent at the fluid level: the states are governed by an action functional which at stationarity is extremized at the self-dual point. The self-duality is a topological property which means that the curvature of a geometrical structure is vanishing. At this point the equation of motion derived from the Lagrangian of the collection of point-like vortices becomes

$$\Delta\psi = \exp(\psi) - 1$$

where ψ is the stream function. We will show that on periodic domains this equation is integrable and we will give exact solutions consisting of arrays of vortices.

4. The exact solution of the vortex dynamics

The inverse scattering method on periodic domains requires the determination of a system of linear differential equations whose compatibility condition is equivalent to the nonlinear equation (Lax pair). We have found for the AH equation (λ is a constant):

$$\begin{pmatrix} -\frac{\lambda^2}{16\sqrt{p}} - \sqrt{p} & -i\frac{\partial}{\partial x} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) \\ i\frac{\partial}{\partial x} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) & -\frac{\lambda^2}{16\sqrt{p}}\exp(u) - \sqrt{p} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0$$

$$\begin{pmatrix} \frac{\lambda^2}{16\sqrt{p}} - \sqrt{p} & -\frac{\partial}{\partial y} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) \\ \frac{\partial}{\partial y} - \frac{1}{4}\left(\frac{\partial u}{\partial y} + i\frac{\partial u}{\partial x}\right) & \frac{\lambda^2}{16\sqrt{p}}\exp(u) - \sqrt{p} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = 0.$$

The possible solutions (ψ_1, ψ_2) must be an intersection of two sets. These equations also determine the spectrum of eigenvalues p where periodic solutions are possible. When a single p has a single eigenfunction (*i.e.* the two eigenfunctions are confounded) $p \equiv p_j$ belongs to the *main spectrum*, and the Wronskian is zero. We assume there are $2N$ such eigenvalues.

We apply the standard procedure, defining the square eigenfunctions f, g and h, and introducing the zeroes of g (the auxiliary spectrum). The solution $\psi(x, y)$ is expressed in terms of these functions. The equations for the auxiliary spectrum are

$$\frac{\partial \gamma_k}{\partial x} = -2\left[\frac{\lambda^2}{16\sqrt{\gamma_k}}\frac{\left(\prod_{l=1}^{N}\gamma_l\right)^2}{\prod_{l=1}^{2N}p_l} + \sqrt{\gamma_k}\right]\frac{\left[\prod_{l=1}^{2N}(\gamma_k - p_l)\right]^{1/2}}{\prod_{\substack{l=1\\l\neq k}}^{N}(\gamma_k - \gamma_l)} \quad (3)$$

$$\frac{\partial \gamma_k}{\partial y} = 2i\left[\frac{\lambda^2}{16\sqrt{\gamma_k}}\frac{\left(\prod_{l=1}^{N}\gamma_l\right)^2}{\prod_{l=1}^{2N}p_l} - \sqrt{\gamma_k}\right]\frac{\left[\prod_{l=1}^{2N}(\gamma_k - p_l)\right]^{1/2}}{\prod_{\substack{l=1\\l\neq k}}^{N}(\gamma_k - \gamma_l)} \quad (4)$$

The solution is based on the fundamental property of $\gamma_k(x, y; p)$ of being defined when p maps the complex plane (of the spectral variable of the Lax operator) to the complex function given by the square root of the Wronskian. Since the later is a polynomial in p, the square root defines a *hyperelliptic Riemann surface*, *i.e.* a compactified double covering of the complex plane with cuts connecting pairs of zeros of the Wronskian. These are the points $\{p_k, k = 1, ..., 2N\}$ of the main spectrum, plus the point zero and the point at infinity. Pairs of zeros p_k are joined by cuts and in addition the origin is connected to infinity. This gives a number of $N + 1$ cuts and generates a compact Riemann surface of genus $g = N$. On this surface there are defined two objects characterising the differential geometry of the curve: (1) a basis of the one dimensional cohomology group of the surface; this means two sets each of N closed paths on the curve (*cycles*), having particular intersection properties. The two sets are noted a_j, and respectively b_j, $j = 1, ..., N$. The intersections are $a_j \circ a_k = 0$, $a_j \circ b_k = \delta_{jk}$ and $b_j \circ b_k = 0$. A typical example, for an elliptic curve $g = 1$ with the topology of the torus, consists of the two possible closed turns around the torus, the short way (a) and the long way (b). (2) A basis in the ring of the one-dimensional differential forms $\mathrm{d}\mu_k = \frac{p^{N-k}dp}{R(p)}$, $k = 1, ..., N$.

With these two sets one calculates several quantities which are invariants of the Riemann surface. Essentially there are calculated integrals of the elements of the basis of differential forms along the cycles a_j and b_j. These are called *periods* and are organized in two matrices

$$A_{ij} = \int_{a_j} \mathrm{d}\mu_i = \int_{a_j} \frac{p^{N-i}\mathrm{d}p}{R(p)},\ i = 1, N,\ j = 1, N$$

$$B_{ij} = \int_{b_j} \mathrm{d}\mu_i = \int_{b_j} \frac{p^{N-i}\mathrm{d}p}{R(p)},\ i = 1, N,\ j = 1, N$$

It is useful to work with the *inverse* of the matrix A, $C = A^{-1}$. Using C, the matrix of A periods is reduced at the identity matrix, and the matrix B becomes $\tau = CB$, the τ-matrix, with positive imaginary part.

Using this geometrical framework, the solution of the γ_k equations can be obtained by operating first a transformation from the set $\{\gamma_k\}$ to a set of functions $\{\phi_k\}$ representing *phases* of motion along the cycles of the Riemann surface. This transformation effectively *linearises* the motion, which can be trivially integrated in these new variables.

We have to define the functions of the target set, the phases $\{\phi_k\}$. They are integrals of linear combinations of the differential one-forms along paths on the Riemann surface, each starting from an initial point γ_0 and ending in the point which corresponds to a function γ_l. The integrand is a combination of the differential one-forms with coefficients from the matrix $C = A^{-1}$

$$\phi_k = -\sum_{l=1}^{N} \int_{\gamma_0}^{\gamma_l} \sum_{m=1}^{N} C_{km} \mathrm{d}\mu_m \qquad (5)$$

The mapping that realizes the correspondence from a collection of points $\{\gamma_l, l = 1, N\}$ of the hyperelliptic Riemann surface to a manifold defined by the collection of points $\{\phi_k, k = 1, N\}$ is called *Abel map*. The manifold generated by the points $\{\phi_k, k = 1, N\}$ has genus $g = N$ (as the initial curve) and has the topology of a torus. It is called *Jacobi torus*. The equations for the phases are

$$\frac{\partial \phi_k}{\partial x} = 2\sum_{m=1}^{N} C_{km} \left[\frac{\lambda^2}{16} \frac{\left(\prod_{n=1}^{N} \gamma_n\right)^2}{\prod_{n=1}^{2N} p_n} \delta(m) + \delta(1-m) \right] = 2C_{k1}$$

$$\frac{\partial \phi_k}{\partial y} = 2i\sum_{m=1}^{N} C_{km} \left[\frac{\lambda^2}{16} \frac{\left(\prod_{n=1}^{N} \gamma_n\right)^2}{\prod_{n=1}^{2N} p_n} \delta(m) - \delta(1-m) \right] = -2iC_{k1}.$$

The equations can be trivially integrated and we obtain the (x, y) dependence of the phases

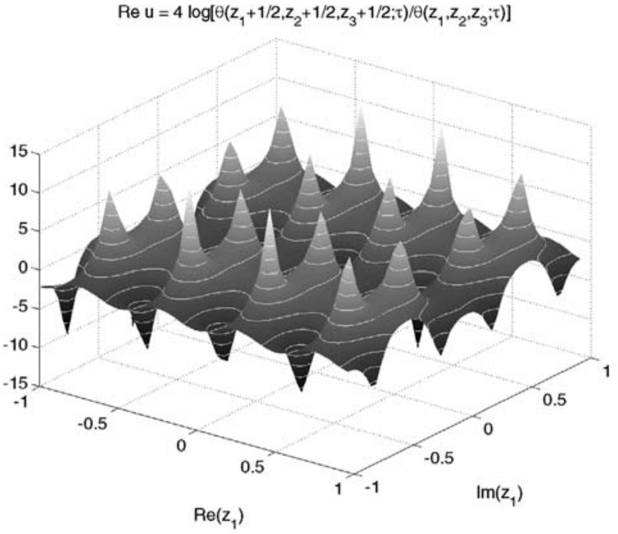

Fig. 2 The exact solution of the stationary Hasegawa-Mima equation, which clearly shows the structure of lattice of vortices.

$$\phi_k(x, y) = 2C_{k1}(x - iy) + \phi_{k0} \qquad (6)$$

where ϕ_{k0} are constants of integration, initial phases.

We note from Eq. (6) that the motion on the Jacobi torus is entirely determined by the main spectrum through the topological properties of the hyeprelliptic Riemann surface (canonical cycles, differential forms, period matrices).

The next step consists of returning to the variables γ_k and from there to the exact solution ψ. This is done as the Jacobi inversion problem. The main role is played by the Θ function. The definition of the Riemann *theta* function involves a vector of dimension N (we denote it by ϕ) and a $N \times N$ matrix τ whose elements have the imaginary part positive.

$$\Theta(\phi, \tau) = \sum_{m_1=-\infty}^{\infty} \cdots \sum_{m_n=-\infty}^{\infty} \exp\left(2\pi i \sum_{k=1}^{N} m_k \phi_k + \pi i \sum_{i=1}^{N} \sum_{j=1}^{N} m_i \tau_{ij} m_j \right).$$

The exact solution is

$$\psi(x, y) = 4\ln\left[\frac{\Theta\left(\phi + \frac{1}{2}\boldsymbol{I}\right)}{\Theta(\phi)} \right] + \text{const}$$

with $\phi_k(x, y) = 2C_{k1}(x - iy) + \phi_{k0}$. A form of the solution is shown in Fig. 2.

5. Conclusion

The intermittent destruction of the regular pattern of the FP solution (zonal flow) exhibits robust vortical structures, strongly suggesting a HM dynamics. As an intersting problem in itself (and for this particular application), the stationary states of the HM eq. need to be investigated analytically. We have shown that these states can be described by a nonlinear model which has a Lax pair of operators and we have obtained (using the inverse scattering method) explicit solutions in terms of the Riemann's *theta* function.

Under the effect of a perturbation this solution evolves to a collection of cuasi-independent vortices with weak interaction. They collide and merge to generate larger structures [4] and the ITG eddies can recover on the linear growth rate scale. The intermittent destruction of the transport barriers traverses these states where the Reynolds stress is isotropic. This supports the intermittent character of the ITB dynamics [7].

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