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A New Method Constructing Magnetic Flux Coordinate

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Abstract

A new algorithm constructing the magnetic flux coordinates is presented. The method is based on the algorithm constructing the generalized magnetic coordinates (GMC), which is a magnetic coordinate without requiring the existence of magnetic surfaces.

Keywords:

magnetic coordinate, flux coordinate, toroidal magnetic configuration

1. Introduction

The magnetic flux coordinates, such as Boozer's coordinates, are the important tool to study the magnetic configurations as well as plasma confinement in toroidal devices. The coordinate is usually constructed by the algorithm given by Boozer [1]; the magnetic surface is constructed by tracing a magnetic line of force for long time, until the line covers the surface sufficiently dense. The two dimensional Fourier transform on the magnetic surface is constructed by one dimensional Fourier transform along the magnetic line of force. This procedure is based upon the assumption that the rotational transform ι is an irrational number, and from the irrational number $m\iota + n = \omega$ two integers, m, and n, are obtained. However, if the rotational transform is near some low order rational number, it is not easy to obtain the reliable results.

The flux coordinates assume the existence of the simply nested magnetic surface structure. In the conventional calculation of the flux coordinates, there is no way to confirm whether the calculated surface is really a surface or not.

The generalized coordinate system (GMC) is introduced as a magnetic coordinate system, which does not require the existence of the nested magnetic surfaces. [2-5] A curvilinear coordinate system (ξ, η, ζ) is the GMC, when the magnetic field $\boldsymbol{B}$ is expressed in the following form

$$\boldsymbol{B} = \Phi(\xi, \eta)\nabla\xi \times \nabla\eta + \nabla\Psi(\xi, \eta, \zeta) \times \nabla\zeta. \qquad (1)$$

Here the variable ζ is the toroidal angle. Note that if the function Φ is allowed to be dependent to ζ, such an expression can be obtained for any curvilinear coordinate system.

The numerical procedure to construct GMC is called as the GMC algorithm. In the procedure the coordinates are deformed so that the ζ-dependence of the vector potential is reduced. When Ψ is a function of only ζ and η, the surface, Ψ = constant, is a magnetic surface. On the contrary, when simply nested structure of magnetic surfaces exists the function Ψ does not depend on the variable ζ.

Since the magnetic surface is not used as a coordinate surface in GMC, the description of the magnetic structure with or without nested magnetic surfaces is possible. However, when the nested structure of magnetic surfaces exists, by utilizing a coordinate surface coinciding with the magnetic surface we can construct the magnetic flux coordinates as a special case of the GMC.

In the next section we consider the GMC and the numerical algorithm to construct GMC. The new method constructing the flux coordinates in case of the existence of the simply nested magnetic surfaces is discussed in Sec. **3**. In the last section the relation with the breaking of the magnetic surfaces is discussed.

2. GMC algorithm

In a curvilinear coordinate system (ξ, η, ζ), the magnetic flux densities H^α are defined as the contravariant component of the magnetic field vector $\boldsymbol{B}$ multiplied by the Jacobian $\sqrt{g}$:

$$H^\xi = \sqrt{g}B^\xi,\ \ H^\eta = \sqrt{g}B^\eta,\ \ H^\zeta = \sqrt{g}B^\zeta. \qquad (2)$$

We assume that the coordinate ζ is a toroidal angle variable with period 2π. If $H^\zeta \equiv \Phi(\xi, \eta)$, then there is a scaler function Ψ, such that

$$H^\xi = \frac{\partial\Psi}{\partial\eta},\ \ H^\eta = -\frac{\partial\Psi}{\partial\xi}, \qquad (3)$$

and the expression (1) is obtained. When the functions H^ξ and H^η depend on ζ, we use H^ξ and H^η instead of Ψ, because the latter include an arbitrariness of the function of ζ.

We use the notations for the averaged part and oscillation

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part with respect to ζ.

$$\overline{A} \equiv \oint A\,\mathrm{d}\zeta \Big/ \oint \mathrm{d}\zeta, \quad \tilde{A} \equiv A - \overline{A}. \tag{4}$$

Then, the condition for GMC can be written as $\tilde{H}^{\zeta} = 0$.

The transformation from GMC to the cylindrical coordinates (r, ϕ, z) is expressed as

$$r = R(\xi, \eta, \zeta) \equiv \xi + \sum_{n \neq 0} R_n(\xi, \eta) \exp(\mathrm{i}\, n\zeta), \tag{5a}$$

$$z = Z(\xi, \eta, \zeta) \equiv \eta + \sum_{n \neq 0} Z_n(\xi, \eta) \exp(\mathrm{i}\, n\zeta), \tag{5b}$$

$$\phi = \zeta. \tag{5c}$$

In this coordinate system, the flux densities are written as

$$H^{\xi} = \frac{\partial Z}{\partial \eta}\left\{B_{\phi}\frac{\partial R}{\partial \zeta} - RB_r\right\} - \frac{\partial R}{\partial \eta}\left\{B_{\phi}\frac{\partial Z}{\partial \zeta} - RB_z\right\}, \tag{6a}$$

$$H^{\eta} = \frac{\partial R}{\partial \xi}\left\{B_{\phi}\frac{\partial Z}{\partial \zeta} - RB_z\right\} - \frac{\partial Z}{\partial \xi}\left\{B_{\phi}\frac{\partial R}{\partial \zeta} - RB_r\right\}, \tag{6b}$$

$$H^{\zeta} = \left\{\frac{\partial Z}{\partial \xi}\frac{\partial R}{\partial \eta} - \frac{\partial Z}{\partial \eta}\frac{\partial R}{\partial \xi}\right\}B_{\phi}. \tag{6c}$$

Equations (6a) and (6b) can be rewritten as

$$\frac{\partial R}{\partial \zeta} + \frac{H^{\xi}}{H^{\zeta}}\frac{\partial R}{\partial \xi} + \frac{H^{\eta}}{H^{\zeta}}\frac{\partial R}{\partial \eta} = \frac{rB_r}{B_{\phi}}, \tag{7a}$$

$$\frac{\partial Z}{\partial \zeta} + \frac{H^{\xi}}{H^{\zeta}}\frac{\partial Z}{\partial \xi} + \frac{H^{\eta}}{H^{\zeta}}\frac{\partial Z}{\partial \eta} = \frac{rB_z}{B_{\phi}}. \tag{7b}$$

These equations are regarded as nonlinear eigenvalue equations for unknown variables R and Z, with eigenvalues $h_{\xi}(\xi, \eta) \equiv H^{\xi}/H^{\zeta}$ and $h_{\eta}(\xi, \eta) \equiv H^{\eta}/H^{\zeta}$.

We solve these nonlinear equations by using the Newtonian method. When the k-th approximation of the coordinates ($R^{(k)}$, $Z^{(k)}$) is known, the flux densities are calculated. The oscillatory potentials $\tilde{a}_{\xi}$ and $\tilde{a}_{\eta}$ are defined by the following relation

$$H^{\xi} = \overline{H}^{\xi} - \frac{\partial \tilde{a}_{\eta}}{\partial \zeta}, \quad H^{\eta} = \overline{H}^{\eta} + \frac{\partial \tilde{a}_{\xi}}{\partial \zeta},$$
$$H^{\zeta} = \overline{H}^{\zeta} + \frac{\partial \tilde{a}_{\eta}}{\partial \xi} - \frac{\partial \tilde{a}_{\xi}}{\partial \eta}. \tag{8}$$

From these relations, we can notice that the reduction of $\tilde{H}^{\zeta}$ can be achieved by the reduction of $\tilde{H}^{\xi}$ and $\tilde{H}^{\eta}$.

The $k+1$-th approximation ($R^{(k+1)}$, $Z^{(k+1)}$) is obtained in order that the oscillatory part of the vector potential is minimized. Putting

$$R^{(k+1)} = R^{(k)} + \Delta R, \quad Z^{(k+1)} = Z^{(k)} + \Delta Z \tag{9}$$

after some algebra, we obtain the relations

$$\Delta R = \frac{1}{\overline{H}^{\zeta}}\frac{\partial R^{(k)}}{\partial \xi}\left\{\frac{\partial \tilde{v}}{\partial \eta} - \tilde{a}_{\eta}\right\} - \frac{1}{\overline{H}^{\zeta}}\frac{\partial R^{(k)}}{\partial \eta}\left\{\frac{\partial \tilde{v}}{\partial \xi} - \tilde{a}_{\xi}\right\}, \tag{10a}$$

$$\Delta Z = \frac{1}{\overline{H}^{\zeta}}\frac{\partial Z^{(k)}}{\partial \xi}\left\{\frac{\partial \tilde{v}}{\partial \eta} - \tilde{a}_{\eta}\right\} - \frac{1}{\overline{H}^{\zeta}}\frac{\partial Z^{(k)}}{\partial \eta}\left\{\frac{\partial \tilde{v}}{\partial \xi} - \tilde{a}_{\xi}\right\}, \tag{10b}$$

where $\tilde{v}$ is a scaler function satisfying the equation

$$\overline{H}^{\xi}\frac{\partial \tilde{v}}{\partial \xi} + \overline{H}^{\eta}\frac{\partial \tilde{v}}{\partial \eta} + \overline{H}^{\zeta}\frac{\partial \tilde{v}}{\partial \zeta} = \overline{H}^{\xi}\tilde{a}_{\xi} + \overline{H}^{\eta}\tilde{a}_{\eta}. \tag{11}$$

Equation (11) is a magnetic differential equation; the periodic solution is obtained only in case that the nested structure of magnetic surfaces exist. In the general case without the nested surfaces eq. (11) is replaced by the weaker condition

$$\delta\iiint_{\Omega}\{\overline{H}^{\xi}\frac{\partial \tilde{v}}{\partial \xi} + \overline{H}^{\eta}\frac{\partial \tilde{v}}{\partial \eta} + \overline{H}^{\zeta}\frac{\partial \tilde{v}}{\partial \zeta} - \overline{H}^{\xi}\tilde{a}_{\xi}$$
$$-\overline{H}^{\eta}\tilde{a}_{\eta}\}^2 = 0. \tag{12}$$

The condition (11) is reduced to an elliptic equation for $\tilde{v}$. The main part of the GMC calculation is consumed in solving the scaler function $\tilde{v}$.

3. GMC algorithm to construct magnetic flux coordinates

In this section we assume the toroidal magnetic configuration with simply nested magnetic surfaces. The region of the nested surfaces with a single magnetic axis is denoted by Ω. For simplicity we assume the last closed surface if supplied as the boundary condition.

In the numerical procedure described in the previous section the $n = 0$ component of the coordinate functions are fixed. If we replace eqs (5a) and (5b) by the following equations

$$r = R(\xi, \eta, \zeta) \equiv \sum_{m,n} R_{m,n}(\xi) \exp[(\mathrm{i}[m\eta + n\zeta)], \tag{13a}$$

$$z = Z(\xi, \eta, \zeta) \equiv \sum_{m,n} Z_{m,n}(\xi) \exp\{\mathrm{i}[m\eta + n\zeta]\}, \tag{13b}$$

and add the requirement that

$$H^{\xi} = 0, \; \frac{\partial}{\partial \eta}\overline{H}^{\eta} = \frac{\partial}{\partial \eta}\overline{H}^{\zeta} = 0, \tag{14}$$

then we can obtain the flux coordinates, ξ being the magnetic surface, and η the poloidal angle variable. This time, we put

$$\overline{H}^{\xi} = \frac{\partial \overline{a}_{\zeta}}{\partial \eta}, \; \overline{H}^{\eta} = \langle \overline{H}^{\eta} \rangle - \frac{\partial \overline{a}_{\zeta}}{\partial \xi}, \; \overline{H}^{\zeta} = \langle \overline{H}^{\zeta} \rangle - \frac{\partial \overline{a}_{\xi}}{\partial \eta}, \tag{15}$$

with $\langle \bar{a}_{\zeta} \rangle = \langle \bar{a}_{\xi} \rangle = 0$, where the brackets stand for the average with respect to η.

The increments of the coordinates are

$$\Delta R = \frac{1}{\overline{H}^{\zeta}}\frac{\partial R^{(k)}}{\partial \xi}\left\{\frac{\partial \tilde{v}}{\partial \eta} + \frac{\partial \overline{v}}{\partial \eta} - \tilde{a}_{\eta}\right\}$$
$$- \frac{1}{\overline{H}^{\zeta}}\frac{\partial R^{(k)}}{\partial \eta}\left\{\frac{\partial \tilde{v}}{\partial \xi} + \frac{\partial \overline{v}}{\partial \xi} - \tilde{a}_{\xi} - \overline{a}_{\xi}\right\}, \tag{16a}$$

$$\Delta Z = \frac{1}{\overline{H}^{\zeta}} \frac{\partial Z^{(k)}}{\partial \xi} \left\{ \frac{\partial \tilde{v}}{\partial \eta} + \frac{\partial \overline{v}}{\partial \eta} - \tilde{a}_{\eta} \right\}$$
$$- \frac{1}{\overline{H}^{\zeta}} \frac{\partial Z^{(k)}}{\partial \eta} \left\{ \frac{\partial \tilde{v}}{\partial \xi} + \frac{\partial \overline{v}}{\partial \xi} - \tilde{a}_{\xi} - \overline{a}_{\xi} \right\}, \qquad (16b)$$

with

$$\overline{H}^{\xi} \frac{\partial \overline{v}}{\partial \xi} + \overline{H}^{\eta} \frac{\partial \overline{v}}{\partial \eta} = \overline{H}^{\xi} \overline{a}_{\xi} + \overline{H}^{\zeta} \overline{a}_{\zeta}. \qquad (17)$$

We can replace the functions $\overline{H}^{\xi}$, $\overline{H}^{\eta}$, and $\overline{H}^{\zeta}$ in the left side of the equation by their averaged values with respect to η. Then we obtain the following equations for $\tilde{v}$, and $\bar{v}$.

$$\langle \overline{H}^{\eta} \rangle \frac{\partial \tilde{v}}{\partial \eta} + \langle \overline{H}^{\zeta} \rangle \frac{\partial \tilde{v}}{\partial \zeta} = \langle \overline{H}^{\eta} \rangle \tilde{a}_{\eta}, \qquad (18a)$$

$$\langle \overline{H}^{\eta} \rangle \frac{\partial \overline{v}}{\partial \eta} = \langle \overline{H}^{\zeta} \rangle \overline{a}_{\zeta}. \qquad (18b)$$

Equation (18a) may have difficulty relating with resonance. For our purpose, by using Fourier expansion

$$\tilde{v} = \sum_{n \neq 0} \sum_{m} (\tilde{v})_{m,n} \exp\{\mathrm{i}(m\eta + n\zeta)\}, \qquad (19a)$$

$$\tilde{a}_{\eta} = \sum_{n \neq 0} \sum_{m} (\tilde{a}_{\eta})_{m,n} \exp\{\mathrm{i}(m\eta + n\zeta)\}, \qquad (19b)$$

eq. (18a) can be solved as

$$(\tilde{v})_{m,n} = -\mathrm{i} \frac{m\iota + n}{(m\iota + n)^2 + \varepsilon^2} \iota (\tilde{a}_{\eta})_{m,n}, \qquad (20)$$

where ι is the rotational transform defined as $\iota(\xi) \equiv \langle \overline{H}^{\eta} \rangle / \langle \overline{H}^{\zeta} \rangle$. Here we introduced small parameter ε (for instance, $\varepsilon = 10^{-6}$) in case that the resonant mode does not vanish on the rational surface.

In the case discussed in the previous section, significant numerical calculation is required to solve the elliptic equation for $\tilde{v}$, because the topology of the magnetic field is not known. On the contrary, in this case, as the simply nested surface structure is assumed, the magnetic differential equation can be solved quite easily.

4. Summary and discussion

In this paper a new method constructing the magnetic flux coordinates is introduced. The method is based on the construction of the generalized magnetic coordinates (GMC). Since the simply nested magnetic surfaces exist only in the asymptotic sense in the asymmetric toroidal magnetic field configuration, there is always a possibility of the magnetic field component destroying the magnetic surfaces. In our method, since the magnetic surfaces are constructed in three-dimensional structure by deforming coordinate surfaces, which is in contrast to the conventional algorithm to obtain surface by covering the surface by a magnetic line of force, any Fourier components on the surface can be calculated. On the rational surfaces the existence of the resonant Fourier mode of H^{ξ} means the breaking of the magnetic surface (emerging of magnetic islands), which can be detected by using our method.

In the region where magnetic surfaces are destroyed, the magnetic field can be written as

$$\boldsymbol{B} = \overline{H}^{\zeta}(\xi) \nabla \xi \times \nabla \eta + \overline{H}^{\eta}(\xi) \nabla \zeta \times \nabla \xi + \boldsymbol{b} \qquad (21)$$

where

$$\boldsymbol{b} = \nabla \tilde{\Psi}(\xi, \eta, \zeta) \times \nabla \zeta. \qquad (22)$$

Reiman and Greenside introduced the "quasi-magnetic" coordinate system to treat the case with magnetic islands or stochastic regions. [6] In their coordinate system, magnetic field has the same form as eq. (21); but the method of construction is completely different. In their method, in the island or stochastic regions the coordinate is calculated by interpolating between the regions with good surfaces, which can be composed by following the magnetic field lines.

The new proposed method dose not mean the economy of the calculation, because to obtain magnetic surfaces in certain precision, it is required for the sufficient number of points to be calculated on the surface.

Detection of the last closed magnetic surfaces is left as the subject of the future works.

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