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Algebraic Behavior of Fluctuations Attributed to Non-Hermitian Property of the Linearized MHD Equation

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Abstract

Algebraic behavior of three-dimensional fluctuations is analyzed for a static equilibrium of ideal magnetohydrodynamics (MHD). Without the reduction of variables to the Lagrange displacement, the linearized ideal MHD equations constitute a non-Hermitian (non-selfadjoint) system. Degeneracy of Alfvén continuous spectra causes a resonant interaction between eigenmodes. Solution of the initial value problem shows a temporally algebraic and spatially localized growth even if there is no exponentially unstable mode.

Keywords:

linear stability, ideal magnetohydrodynamics, Alfvén continuous spectrum, algebraic behavior, non-Hermitian operator

1. Introduction

The algebraic instability has been attracting a lot of interest in the linear stability analysis of various fluids and plasmas (non-neutral plasma, galactic phenomena, atmospheric dynamics and so on). Since the growth of the perturbation field is temporally algebraic ($\propto t^{\alpha}$), this instability can not be found by the conventional normal modes analysis or the dispersion relation which assumes the exponential behavior ($e^{-i\omega t}$ with eigenvalue ω) of solutions and the existence of a complete set of eigenfunctions.

If we write linearized equations in the form of an evolution equation

$$\mathrm{i}\partial_t f = \mathcal{K} f, \tag{1}$$

the spectral resolution of the linear operator $\mathcal{K}$ is guaranteed by the Von Neumann theorem so far as $\mathcal{K}$ is Hermitian (selfadjoint). Then the solution f can be completely decomposed into eigenfunctions (or eigenmodes) which may be singular if $\mathcal{K}$ has the continuous spectrum. The algebraic behavior stems from the non-Hermitian property of $\mathcal{K}$. A typical example is the Orr-Sommerfeld and Squire equations [1] which are known as the incompressible viscid fluid equations linearized around laminar shear flow. Its mathematical structure is written like

$$\mathrm{i}\partial_t \begin{pmatrix} \tilde{w}_x \\ \Delta \tilde{v}_x \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{sq} & \mathcal{K}_c \\ 0 & \mathcal{K}_{os} \end{pmatrix} \begin{pmatrix} \tilde{w}_x \\ \Delta \tilde{v}_x \end{pmatrix}, \tag{2}$$

where $\tilde{v}_x$ and $\tilde{w}_x$, respectively, denote the x component of velocity and vorticity perturbation. If the spectrum of $\mathcal{K}_{sq}$ coincides with that of $\mathcal{K}_{os}$ (degeneracy), we observe linear growth ($\propto t$) of $\tilde{w}_x$ due to the resonance between eigenmodes, which is analogous to the Jordan canonical form of the non-Hermitian matrix in linear algebra. This algebraic growth is expected to reach nonlinear regime and trigger turbulence.

However, the spectral resolution of non-Hermitian operators is mathematically unsolved, and therefore we must solve the initial value problem directly. Especially in ideal fluids and plasmas, the continuous spectrum reflects the infinite dimensional property of functional space which cannot be understood as a simple extension of linear algebra [2]. It is a mathematically nontrivial problem what happens if the resonance between singular eigenmodes occurs.

In this short paper, we will demonstrate an algebraic behavior which occurs in ideal magnetohydrodynamics (MHD) with slab geometry. It will be produced by degeneracy of Alfvén continuous spectra. The algebraic growth is localized on the rational surface and may ubiquitously occur even if no exponential instability exists.

2. Non-Hermicity of ideal plasma

In general, stability of plasma is analyzed by solving the linearized MHD equations (simultaneous partial differential equations), which is represented by an evolution equation for the fluctuation part of velocity ($\tilde{\boldsymbol{v}}$), magnetic field ($\tilde{\boldsymbol{b}}$) and pressure ($\tilde{p}$). In many cases, these perturbation fields in the vicinity of the equilibrium are reduced to a displacement vector ($\tilde{\boldsymbol{\xi}}$) and the linearized equation is written as $\partial_t^2 \tilde{\boldsymbol{\xi}} = \mathcal{H}\tilde{\boldsymbol{\xi}}$ where $\mathcal{H}$ is known as a Hermitian operator [3]. Since the spectral resolution of Hermitian operator is

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mathematically possible, the MHD stability analysis is conventionally based on the dispersion relation which determines if the imaginary part of ω is positive or not.

However, the special initial conditions, $\tilde{p}|_{t=0} = 0$ and $\tilde{\boldsymbol{b}}|_{t=0} = 0$, are assumed in the above reduction process, and the spectrum of the original system (1) for the seven variables $f = (\tilde{\boldsymbol{v}}, \tilde{\boldsymbol{b}}, \tilde{p})$ does not perfectly coincide with that of $\mathcal{H}$.

In the following analysis, let us assume the incompressibility for simplicity and consider a slab equilibrium,

$$\boldsymbol{B} = (0, B_y(x), B_z(x)), \quad \boldsymbol{V} = (0,0,0), \tag{3}$$

$$P(x) + |\boldsymbol{B}(x)|^2/2 = \text{const.} \tag{4}$$

Taking strong magnetic shear into account, we do not adopt the reduced MHD approximation, but consider "three-dimensional" fluctuations, i.e.,

$$\tilde{\boldsymbol{v}}(x,t)\mathrm{e}^{i(k_y y + k_z z)}, \tag{5}$$

$$\tilde{\boldsymbol{b}}(x,t)\mathrm{e}^{i(k_y y + k_z z)}, \tag{6}$$

$$\tilde{p}(x,t)\mathrm{e}^{i(k_y y + k_z z)}, \tag{7}$$

where we made use of the wavenumber k_y and k_z. We will also use $\boldsymbol{k} = (0, k_y, k_z)$ and $k = |\boldsymbol{k}|$ for simple notation. Let us introduce

$$\tilde{w}_x = \mathrm{i}k_y \tilde{v}_z - \mathrm{i}k_z \tilde{v}_y, \tag{8}$$

$$\tilde{j}_x = \mathrm{i}k_y \tilde{b}_z - \mathrm{i}k_z \tilde{b}_y, \tag{9}$$

as independent variables. Due to the incompressibility, the pressure fluctuation $\tilde{p}$ is given by the Poisson equation and the linearized system can be described by the four independent variables, $\tilde{v}_x$, $\tilde{w}_x$, $\tilde{b}_x$ and $\tilde{j}_x$ as follows.

$$\mathrm{i}\partial_t \tilde{w}_x = -\omega_a(x)\tilde{j}_x + \iota(x)\tilde{b}_x, \tag{10}$$

$$\mathrm{i}\partial_t \tilde{j}_x = -\omega_a(x)\tilde{w}_x - \iota(x)\tilde{v}_x, \tag{11}$$

$$\mathrm{i}\partial_t \Delta\tilde{v}_x = -\omega_a(x)\Delta\tilde{b}_x + \omega_a''(x)\tilde{b}_x, \tag{12}$$

$$\mathrm{i}\partial_t \tilde{b}_x = -\omega_a(x)\tilde{v}_x, \tag{13}$$

where $\Delta = \partial_x^2 - k^2$ is Laplacian operator, a prime (′) denotes x derivative, $\omega_a(x) = \boldsymbol{k}\cdot\boldsymbol{B}(x)$ denotes local Alfvén frequency and we put $\iota(x) = [\boldsymbol{B}'(x)\times\boldsymbol{k}]\cdot\mathbf{e}_x$. In terms of $\mathcal{M}_\pm = \Delta(\tilde{v}_x \mp \tilde{b}_x)$ and $\mathcal{S}_\pm = \tilde{w}_x \mp \tilde{j}_x$, these equations are rewritten in the matrix form of

$$\mathrm{i}\partial_t \begin{pmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{M}_+ \\ \mathcal{M}_- \end{pmatrix} = \begin{pmatrix} \omega_a & 0 & 0 & \iota\Delta^{-1} \\ 0 & -\omega_a & -\iota\Delta^{-1} & 0 \\ 0 & 0 & \omega_a + \omega_a'\partial_x\Delta^{-1} & (\omega_a'\partial_x + \omega_a'')\Delta^{-1} \\ 0 & 0 & -(\omega_a'\partial_x + \omega_a'')\Delta^{-1} & -\omega_a - \omega_a'\partial_x\Delta^{-1} \end{pmatrix} \begin{pmatrix} \mathcal{S}_+ \\ \mathcal{S}_- \\ \mathcal{M}_+ \\ \mathcal{M}_- \end{pmatrix}. \tag{14}$$

If we consider normal modes such as $\tilde{v}_x(x,t) = \tilde{v}_x(x)\mathrm{e}^{-i\omega t}$, (12) and (13) are combined into, by eliminating $\tilde{b}_x$,

$$\partial_x\left[\left(\omega^2 - \omega_a^2\right)\partial_x \tilde{v}_x\right] - k^2\left(\omega^2 - \omega_a^2\right)\tilde{v}_x = 0, \tag{15}$$

which is well known as the eigenvalue problem of the Alfvén wave (the stream function replaces $\tilde{v}_x$ in the case of two-dimensional fluctuation). Barston [4] considered this problem and proved that there is no spectrum in addition to the Alfvén continuous spectra σ_c^+ and σ_c^-, where

$$\sigma_c^\pm = \{\omega \in \mathbb{R}; \omega = \pm\omega_a(x)\}. \tag{16}$$

Furthermore, the upper two equations in (14) have the same continuous spectra due to the multiplication operator $\pm\omega_a(x)$. As a result, the evolution equation has four degenerate continuous spectra in total. Since the variables $\mathcal{S}_\pm$ are forced by $\mathcal{M}_\pm$ like the Orr-Sommerfeld and Squire equations, we can expect an algebraic growth of $\mathcal{S}_\pm$ due to the resonance between the continuous spectra. But, since we do not have mathematical theory to describe it, we will solve the initial value problem in the next section.

3. Analysis of algebraic behavior

We assume that the profile of the ambient magnetic field is linear and the rational surface is located on $x = 0$;

$$\omega_a(x) = \boldsymbol{k}\cdot\boldsymbol{B}(x) = x. \tag{17}$$

In this coordinate system, we consider a finite domain $x \in [-L_1, L_2]$ ($L_1 > 0$, $L_2 > 0$) with a Dirichlet boundary condition

$$\tilde{v}_x(-L_1,t) = \tilde{v}_x(L_2,t) = 0, \tag{18}$$

$$\tilde{b}_x(-L_1,t) = \tilde{b}_x(L_2,t) = 0. \tag{19}$$

We denote the spectrum by $\sigma_c = \sigma_c^+ \cup \sigma_c^-$ which are nothing but $\sigma_c^+ = [-L_1, L_2]$ and $\sigma_c^- = [-L_2, L_1]$. Using the Laplace transformation defined by

$$\mathcal{L}[\tilde{v}_x(x,t)] = \int_0^\infty \tilde{v}_x(x,t)\mathrm{e}^{i\Omega t}\mathrm{d}t, \ (\Omega \in \mathbb{C}) \tag{20}$$

we obtain, instead of (15),

$$\partial_x\left[(\Omega^2 - x^2)\partial_x\tilde{V}\right] - k^2(\Omega^2 - x^2)\tilde{V} = \mathrm{i}\Omega\tilde{v}_x(x,0) - \mathrm{i}x\Delta\tilde{b}_x(x,0), \tag{21}$$

where $\tilde{V}(x,\Omega) = \mathcal{L}[\tilde{v}_x(x,t)]$. If the initial conditions $\tilde{v}_x(x,0)$ and $\tilde{b}_x(x,0)$ are regular functions, this solution $\tilde{V}(x,\Omega)$ becomes singular at $\{(x,\Omega); x = \pm\Omega\}$. For $\Omega = \omega \in \sigma_c\backslash\{0\}$, there are two singular points $x = \pm\omega$, and the Frobenius method gives the well known logarithmic singularities, $\log(x \pm \Omega)$. The inverse Laplace transformation of these singularities shows the phase mixing damping in proportion to $1/t$ [4,5].

On the other hand, for $\Omega = 0$, the two singular points overlap at the rational surface $x = 0$. After some careful considerations, we can write the general solution around $x = \Omega = 0$ as

$$\tilde{V}(x,\Omega) = C_1(\Omega) + C_2(\Omega)\log\frac{x-\Omega}{x+\Omega} + O(\Omega), \qquad (22)$$

where $C_1(\Omega)$ and $C_2(\Omega)$ are coefficients determined by the outer solution and the boundary condition (so-called continuation data), and $\mathcal{O}(\Omega)$ denotes the terms that vanish when $\Omega \to 0$. For the limit of $\Omega \to \pm i0$, we obtain

$$\tilde{V}(x,\pm i0) = C_1(\pm i0) \mp C_2(\pm i0)2\pi i Y(-x), \qquad (23)$$

where $Y(x)$ denotes the Heaviside function. In the domain $[-L_1, L_2]$ except for $x = 0$, $\tilde{V}(x, \pm i0)$ must satisfy (21) with $\Omega = \pm i0$, i.e.,

$$\partial_x\left[x^2\partial_x\tilde{V}\right] - k^2x^2\tilde{V} = ix\Delta\tilde{b}_x(x,0). \qquad (24)$$

Solving this equation and continuating the solution with (23) result in

$$\tilde{V}(x,\pm i0) = i\frac{\tilde{b}_x(x,0) - \tilde{b}_x(0,0)g(x)}{x}, \qquad (25)$$

where

$$g(x) = \begin{cases} \dfrac{\sinh(-kL_1 - kx)}{\sinh(-kL_1)} & x\in[-L_1, 0] \\ \dfrac{\sinh(kL_2 - kx)}{\sinh(kL_2)} & x\in[0, L_2] \end{cases}. \qquad (26)$$

From (13), $\tilde{B}(x,\Omega) = \mathcal{L}[\tilde{b}_x(x,t)]$ is given by

$$\tilde{B}(x,\Omega) = \frac{-x\tilde{V}(x,\Omega) + i\tilde{b}_x(x,0)}{\Omega}. \qquad (27)$$

By the inverse Laplace transformation, we can finally obtain the asymptotic behavior ($t \to \infty$) as

$$\tilde{v}_x(x,t) \to 0, \qquad (28)$$

$$\tilde{b}_x(x,t) \to \tilde{b}_x(0,0)g(x). \qquad (29)$$

This convergence speed is characterized by $1/t$.

The behavior of the remaining variables, $\tilde{w}_x$ and $\tilde{j}_x$, are easily estimated by using the above result. First, we obtain, by definition,

$$\Delta^{-1}\mathcal{L}[\mathcal{M}_\pm] = \frac{(\Omega \pm x)\tilde{V}(x,\Omega) \mp i\tilde{b}_x(x,0)}{\Omega}. \qquad (30)$$

By substituting this expression, the Laplace transformation of the upper two equations in (14) leads to

$$\mathcal{L}[\mathcal{S}_\pm] = \pm\frac{\iota(x)}{\Omega}\left[\tilde{V}(x,\Omega) \pm \frac{i\tilde{b}_x(x,0)}{\Omega \mp x}\right] + \frac{i\mathcal{S}_\pm(x,0)}{\Omega \mp x}. \qquad (31)$$

The last term in the right hand side yields an solution which shows a phase mixing like $\mathcal{S}_\pm(x,0)e^{\mp ixt}$. On the other hand, the singularity of $1/\Omega(\Omega \mp x)$ brings about a localized algebraic growth at $x = 0$ as far as $\iota(0) \neq 0$ and $\tilde{b}_x(0,0) \neq 0$, which can be written like

$$\mathcal{S}_\pm \sim \pm\iota(0)\tilde{b}_x(0,0)\frac{1-e^{\mp ixt}}{x}. \qquad (32)$$

By calculating $\tilde{w}_x = (\mathcal{S}_+ + \mathcal{S}_-)/2$ and $\tilde{j}_x = (\mathcal{S}_- - \mathcal{S}_+)/2$, we can find that $\mathrm{Im}(\tilde{w}_x)$ and $\mathrm{Re}(\tilde{j}_x)$ increase in proportion to time around $x = 0$. Since we supposed $\boldsymbol{k}\cdot\boldsymbol{B}(0) = 0$, $\mathrm{Im}(\tilde{w}_x)$ corresponds to the parallel component of $\tilde{\boldsymbol{v}}$ with respect to $\boldsymbol{B}$; $\tilde{v}_\parallel(0,t) \propto t$.

4. Summary

By considering incompressible three-dimensional fluctuations, the linearized ideal MHD equation has four Alfvén continuous spectra. They degenerate with each other and the resonant interaction occurs only at zero frequency due to the non-Hermitian property of the evolution equation. Although the spectral theory for non-Hermitian operator has not been established, the initial value problem indicates the existence of algebraic growth which is localized on the rational surface.

On the rational surface, the initial perturbation of magnetic field receives no force since it does not bend the field line. Then, the field line remains inclined with respect to the ambient pressure contour, which causes a constant acceleration of parallel plasma motion; $\tilde{v}_\parallel \propto t$. This physical mechanism of the algebraic instability is not included in the stability analysis of the Lagrange displacement which assumes $\tilde{b}_x(x,0) \equiv 0$.

In more realistic situations, the viscosity and the resistivity have a considerable effect on the fine structure that we showed. However, our algebraic instability induces the growth of perturbation energy in small scale, which can rapidly grow depending on the initial condition, and may be related to the development of turbulence in inhomogeneous magnetic field. The effect of shear flow will be discussed elsewhere.

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