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Relaxation with High-Speed Plasma Flows and Singularity Analysis in MHD Equilibrium

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Abstract

Relaxation model that leads to plasma confinement with rigid-rotation is presented. This model applies to Jupiter's magnetosphere. It is shown that the invariance of canonical angular momentum of electron fluid, which is realized by axisymmetry through self-organization process, yields plasma confinement. Including poloidal flows in equilibrium equation makes the problem rather complicated. Singularity due to the poloidal flow is focused on. It is shown that the singular equation for equilibrium has the same structure as the equation for linear Alfvén wave. Since the singular solution for equilibrium equation is physically inadequate, the singularity may be removed by another physical effect. The Hall-effect is taken into account as a singular perturbation that removes the singularity of equilibrium equation for ideal magnetohydrodynamics.

Keywords:

relaxation, Jupiter, equilibrium, singularity, singular perturbation

1. Introduction

We construct a relaxation model that leads to plasma confinement with rigid-rotation. This research is motivated by the observation that shows high-β plasma with rigid-rotation in Jupiter's magnetosphere [1]. Since previous theoretical models for Jupiter's magnetosphere are based on the stationary solution of the ideal magnetohydrodynamics (MHD) equations, flow and pressure profiles are arbitrary. To exclude these arbitrariness, we invoke relaxation theory. Although it is known that a relaxed state with rigid-rotation is achieved by imposing the constraint on total angular momentum [2], the plasma cannot be confined because of the centrifugal force. To confine the plasma, we must impose rather restricted condition: axisymmetry of magnetic field during relaxation process, which assumes the constancy of electron's canonical angular momentum. With that assumption, it is shown that the plasma can be confined by the Lorentz force arising from the difference between angular velocities of electron and ion fluids.

Since the above equilibrium has only toroidal flow (rigid-rotation), the equation has the same structure as the Grad-Shafranov (GS) equation for static equilibrium. Including poloidal flows requires to generalize the GS equation. It is well known that the ideal-MHD equilibrium with incompressible flows has a singularity where poloidal flow coincides with Alfvén speed defined by poloidal magnetic field [3]. By using one-dimensional model, it is shown that this singularity has the same structure as the equation for linear Alfvén wave. It is expected that the singularity is removed by another physical effect because singular solution for the equilibrium problem may be physically inadequate. It has been shown that the Hall-effect can remove the singularity in the equation for linear Alfvén wave [4]. We will discuss the possibility of Hall-effect to remove the singularity in the equation for equilibrium.

2. Relaxation with high-speed plasma flows

We consider a plasma that obeys ideal-MHD equations in an axisymmetric domain Ω. The ideal-MHD equations, with standard notation in Alfvén units, are

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{V} \times \boldsymbol{B}) = 0, \tag{1}$$

$$\partial_t \boldsymbol{V} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla p, \tag{2}$$

where the incompressible flow is assumed, $\nabla \cdot \boldsymbol{V} = 0$. With the boundary condition $\boldsymbol{n} \cdot \boldsymbol{B} = 0$, $\boldsymbol{n} \cdot \boldsymbol{V} = 0$ (on $\partial\Omega$), where $\boldsymbol{n}$ is a unit normal vector onto $\partial\Omega$, the basic eqs. (1) and (2) have the following global invariants; the total energy $E = 0.5\int_\Omega (V^2 + B^2)dx$, the magnetic helicity $H_1 = 0.5\int_\Omega \boldsymbol{A} \cdot \boldsymbol{B} dx$ and the cross helicity $H_2 = \int_\Omega \boldsymbol{V} \cdot \boldsymbol{B} dx$. Moreover, we have an invariant that stems from the toroidicity of the domain, that is, total angular momentum. In cylindrical coordinate r-θ-z, the total angular momentum is defined as follows

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$$M = \int_{\Omega} r\boldsymbol{e}_\theta \cdot \boldsymbol{V} d\,x. \tag{3}$$

If $\boldsymbol{n} \cdot \boldsymbol{e}_\theta = 0$ on $\partial\Omega$ is satisfied, M is conserved.

Using the above invariants, we apply the self-organization theory. It should be noted that E or H_2 cannot be selected as the target functional (TF) to be minimized because both are equivalent in their fragility. The correct procedure is to search for the most smooth solution in the norm of the vorticities [5]. The TF should be $F = 0.5\int_\Omega (B^2 + |\nabla \times \boldsymbol{V}|^2)dx$. Minimizer of F under the constraint on E, H_1, H_2 and M is given by the variational principle

$$\delta(F - \mu_0 E - \mu_1 H_1 - \mu_2 H_2 - \omega M) = 0, \tag{4}$$

where μ_j (j = 0, 1, 2) and ω are Lagrange multipliers. Variations of $\delta\boldsymbol{A}$ and $\delta\boldsymbol{V}$, under the boundary conditions $\boldsymbol{n} \times \delta\boldsymbol{A} = 0$ and $\boldsymbol{n} \times \delta\boldsymbol{V} = 0$, yield the following Euler-Lagrange equations

$$(1 - \mu_0)\, \nabla \times \boldsymbol{B} - \mu_1 \boldsymbol{B} - \mu_2 \nabla \times \boldsymbol{V} = 0, \tag{5}$$

$$\nabla \times (\nabla \times \boldsymbol{V}) - \mu_0 \boldsymbol{V} - \mu_2 \boldsymbol{B} - r\omega \boldsymbol{e}_\theta = 0. \tag{6}$$

The general solution is given by a triple Beltrami field with vertical field and with rigid-rotation for $\boldsymbol{V}$

$$\boldsymbol{V} = \sum_{j=1}^{3} D_j \boldsymbol{G}_j - \frac{2\mu_2^2\omega}{\mu_0^2\mu_1}\boldsymbol{e}_z - r\frac{\omega}{\mu_0}\boldsymbol{e}_\theta, \tag{7}$$

$$\boldsymbol{B} = \sum_{j=1}^{3} \frac{1}{\mu_2}\left(\lambda_j^2 - \mu_0\right) D_j \boldsymbol{G}_j + \frac{2\mu_2\omega}{\mu_0\mu_1}\boldsymbol{e}_z, \tag{8}$$

where $\nabla \times \boldsymbol{G}_j = \lambda_j \boldsymbol{G}_j$. These solutions do not even satisfies the boundary condition (because of the vertical field) nor represent equilibrium. Next, we consider "adjustment process," which reads

$$\delta(E - \mu_1' H_1 - \mu_2' H_2 - \omega' M) = 0, \tag{9}$$

where μ_1', μ_2' and ω' are constants determined by variational principle (4). Euler-Lagrange equations for the formal variational principle (9) are

$$\nabla \times \boldsymbol{B} - \mu_1' \boldsymbol{B} - \mu_2' \nabla \times \boldsymbol{V} = 0, \tag{10}$$

$$\boldsymbol{V} - \mu_2' \boldsymbol{B} - r\omega' \boldsymbol{e}_\theta = 0, \tag{11}$$

which gives

$$\boldsymbol{B} = C_1 \boldsymbol{G}_1 - \frac{2\mu_2'\omega'}{\mu_1'}\boldsymbol{e}_z, \tag{12}$$

$$\boldsymbol{V} = \mu_2' C_1 \boldsymbol{G}_1 - \frac{2\mu_2'^2\omega'}{\mu_1'}\boldsymbol{e}_z + r\omega'\boldsymbol{e}_\theta, \tag{13}$$

where $\nabla \times \boldsymbol{G}_1 = \frac{\mu_1'}{1-\mu_2'^2}\boldsymbol{G}_1$. The state expressed by eqs. (12) and (13), which represent an equilibrium state, is a special class of eqs. (7) and (8). Furthermore, to satisfy the boundary condition, $\mu_2' = 0$ or $\omega' = 0$ must be required. If $\omega' = 0$, the solutions reduce to the well-known single-Beltrami field with a parallel flow. To obtain rigid-rotation, we put $\mu_2' = 0$, which yields force-free magnetic field and rigid-rotation. The magnetic and the flow fields are decoupled. In this case, the Bernoulli law reads

$$p - r^2\omega'^2/2 = \text{const}, \tag{14}$$

which means the pressure rises toward the plasma edge due to the centrifugal force.

To confine the plasma, we will consider rather restricted case that the magnetic field is axisymmetric through relaxation process (allowing the representation $\boldsymbol{B} = \nabla\Psi(r, z, t) \times \nabla\theta + rB_\theta(r, z, t)\nabla\theta$). This assumption may be justified by the observation that the magnetic field of Jupiter's inner magnetosphere is almost axisymmetric dipole [1]. Then, the canonical angular momentum of electron fluid (or the total poloidal flux)

$$P = -\int_\Omega r\boldsymbol{e}_\theta \cdot \boldsymbol{A} dx = -\int_\Omega \Psi dx \tag{15}$$

is conserved because $dP/dt = \int_{\partial\Omega} \boldsymbol{n} \cdot (\Psi \boldsymbol{V}) ds = 0$.

Relaxed state derived by similar scenario reads

$$\boldsymbol{V} = r\omega'\boldsymbol{e}_\theta, \tag{16}$$

$$-\Delta^*\Psi - \mu_1'^2\Psi + r^2\omega_1' = 0, \tag{17}$$

where $\Delta^* = r\partial_r r^{-1}\partial_r + \partial_z^2$ is the usual Grad-Shafranov operator and ω_1' is a constant related to P. The Bernoulli's law takes the form as

$$p - r^2\omega'^2/2 + \omega_1'\Psi = \text{const}. \tag{18}$$

The third term in the LHS of eq. (18) is the crucial effect compared with eq. (14). This term stems from the conservation of electron's canonical angular momentum and represents the Lorentz force arising from toroidal current yielded by the different angular velocities of electron and ion fluids. Typical solutions for the pressure profiles based on eq. (14) and eq. (18) are shown in Figs. 1 and 2, respectively. Figure 1 shows that the pressure rises toward the plasma edge, while Fig. 2 shows that the pressure *falls*. From Fig. 2, it can be said that axisymmetry is needed to confine the plasma and that this model can apply to the qualitative explanation of the phenomena in Jupiter's magnetosphere.

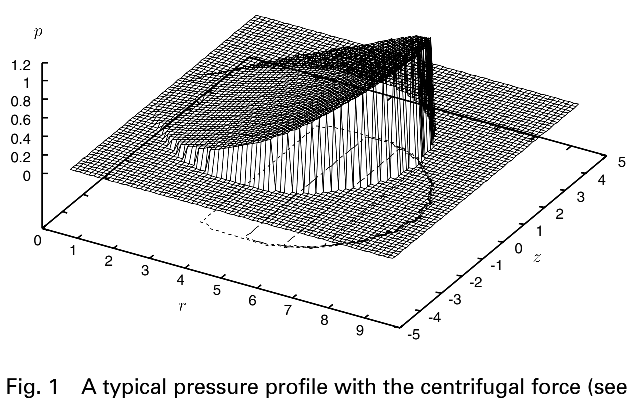

Fig. 1 A typical pressure profile with the centrifugal force (see eq. (14)).

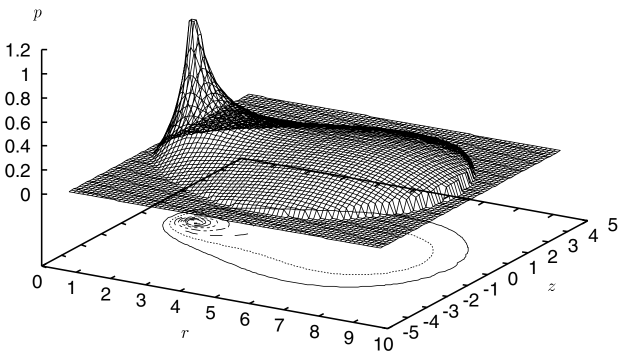

Fig. 2 A typical pressure profile for axisymmetric relaxation (See eq. (18)).

3. Singularity in MHD equilibria

Since the relaxed state obtained in the previous section has only toroidal flow (rigid-rotation), the governing equation (17) has the same structure as the GS equation. Including poloidal flow makes the equilibrium problem rather complicated. Assuming the incompressible flow, we use $\boldsymbol{V} = \nabla\Phi(x, y) \times \nabla z + V_z(x, y)\nabla z$ and $\boldsymbol{B} = \nabla\Psi(x, y) \times \nabla z + B_z(x, y)\nabla z$ in Cartesian coordinate. The equation for ideal-MHD equilibria can be formulated as

$$(1-M^2(\Phi))\,\Delta\Phi - \frac{1}{2}\dot{M}^2(\Phi)\,|\nabla\Phi|^2 + \dot{H}(\Phi) = 0, \qquad (19)$$

where the dot denotes derivative with respect to Φ, $M^2(\Phi) = |\nabla\Phi|^2/|\nabla\Psi|^2 = 1/\dot{\Psi}^2$ is the poloidal Alfvén Mach number and $H(\Phi) = p + |\nabla\Phi|^2/2 + B_z^2/2$. Equation (19) becomes singular at the point $M^2 = 1$. We examine the physical meaning of the singularity. Assuming one-dimensional system (depending on only x) and putting $\Phi = \Phi_0 + \phi(|\phi| \ll |\Phi_0|)$, we can linearize eq. (19) as

$$\frac{d\Phi_0}{dx}\frac{d}{dx}\left[(1-\Psi^2(\Phi_0))\frac{d\phi}{dx}\right] - \left\{\frac{d}{dx}\left[(1-\Psi^2(\Phi_0))\frac{d^2\Phi_0}{dx^2}\right]\right\}\phi = 0. \qquad (20)$$

On the other hand, the equation for linear Alfvén wave has a form of

$$\frac{d}{dx}\left[\left(\Omega^2 - k^2B_y^2\right)\frac{d}{dx}\left(\frac{\phi}{\Omega}\right)\right] - k^2\left(\Omega^2 - k^2B_y^2\right)\left(\frac{\phi}{\Omega}\right) = 0, \qquad (21)$$

where ϕ is the stream function of the fluctuation of the flow and $\Omega = \omega - kV_y$ is the Doppler-shifted frequency [6]. Putting $\tilde{\omega} = \omega/k = 0$ (equilibrium) with $k \neq 0$, eq. (21) becomes

$$V_y\frac{d}{dx}\left[\left(1-B_y^2/V_y^2\right)\frac{d\phi}{dx}\right] - \left\{\frac{d}{dx}\left[\left(1-B_y^2/V_y^2\right)\frac{dV_y}{dx}\right]\right\}\phi - k^2\left(1-B_y^2/V_y^2\right)V_y\phi = 0. \qquad (22)$$

Since we can write $d\Phi_0/dx = -V_y$ and $\Psi^2(\Phi_0) = B_y^2/V_y^2$ in eq. (20), eq. (22) is identical to the principal part of eq. (20). For linear Alfvén wave, it has been shown that the Hall-effect can change the continuous spectrum due to the singularity [4]. Thus, we consider the Hall-effect as a candidate to remove the singularity.

Hall-MHD is described by eq. (2) and

$$\partial_t\boldsymbol{B} - \nabla \times [(\boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B}) \times \boldsymbol{B}] = 0 \qquad (23)$$

with $\nabla\cdot\boldsymbol{V} = 0$ and $\nabla\cdot\boldsymbol{B} = 0$. The coefficient ε is the measure of the ion skin depth $l_i = \sqrt{M/\mu_0 ne^2}$ (M is the ion mass). Equilibrium model that is based on Hall-MHD and that reproduces ideal-MHD equilibrium with the limit $\varepsilon \to 0$ has been obtained by Ilgisonis [7]. Ilgisonis's model has two characteristic directions that degenerate in the limit $\varepsilon \to 0$, e.g., $\boldsymbol{V}$ and $\boldsymbol{V}_e = \boldsymbol{V} - \varepsilon\nabla \times \boldsymbol{B}$. Writing $\boldsymbol{V}_e = \nabla\Phi_e(x, y) \times \nabla z + V_{ez}(x, y)\nabla z$ and $\boldsymbol{B}_* = \boldsymbol{B} + \varepsilon\nabla \times \boldsymbol{V} = \nabla\Psi_*(x, y) \times \nabla z + B_{*z}\nabla z$, equilibrium equations become

$$\Delta\Phi - \Psi'^2\Delta\Phi_e - \Psi'\Psi''|\nabla\Phi_e|^2 + \dot{F} + V_{ez}\partial_\Phi V_{ez} - \delta\phi_s/\varepsilon = 0, \qquad (24)$$

$$\Phi_e = \Phi - \varepsilon B_z,\ B_z = V_{ez}\Psi' - \phi_s', \qquad (25)$$

$$V_{ez} = V_z + \varepsilon\,(\Psi'\Delta\Phi_e + \Psi''|\nabla\Phi_e|^2), \qquad (26)$$

where the prime denotes $d/d\Phi_e$, $V_z = V_z(\Phi, B_z)$ and ϕ_s is the scalar potential. Here, the notation $\phi_s|_{\Phi_e\to\Phi}$ means that the argument of function $\phi_s(\Phi_e)$ is formally changed by Φ, and the notation $\delta\phi_s = \phi|_{\Psi\to\Psi_*} - \phi(\Psi)$ and $F = p + V^2/2 - \phi_s|_{\Phi_e\to\Phi}/\varepsilon$ is defined. Assuming $\varepsilon \ll 1$, we can expand eq. (24) and the principal part becomes

$$\left(1 - M_{\mathrm{I}}^2(\Phi) - \varepsilon M_{\mathrm{I}}^{2\prime}(\Phi)B_z + \varepsilon^2(\cdots)\right)\Delta\Phi + \varepsilon^2 M_{\mathrm{I}}^2(\Phi)\Delta(\Delta\Phi) + \cdots = 0, \qquad (27)$$

where $M_{\mathrm{I}}^2(\Phi) = 1/\Psi'^2|_{\Phi e\to\Phi}$ is the poloidal Alfvén Mach number in the limit $\varepsilon \to 0$. Although the term with the highest (fourth) derivative contains the coefficient M_{I}^2, singularity is removed as seen in eq. (27) because $M_{\mathrm{I}}^2 \neq 0$ is assumed in the present theory.

4. Summary

The essential point in the relaxation model presented in Sec. 2 is the axisymmetry of the magnetic field through relaxation process. Under this restricted condition, the canonical angular momentum of electron fluid is conserved. In this case, the equilibrium equation for the relaxed state is slightly modified compared with the case of total angular momentum constraint. Due to the difference of angular velocity, the resultant Bernoulli's law includes Lorentz force, and the plasma confinement is achieved. On the other hand, the relaxed state of the ion and electron fluids with the same angular velocity is related to a force-free magnetic field, and the plasma is not confined because of the centrifugal force.

The physical meaning of the singularity in ideal-MHD equilibrium equation is examined. By using one-dimensional model, the structure of the singular equation is shown to be principally identical to the equation for linear Alfvén wave.

The solution may be expected to be singular, which is physically unrealizable. This singularity is expected to be removed by another mechanism, e.g., Hall-effect. In this point of view, the Hall effect as a singular perturbation is discussed. The Hall-effect imposes rather complicated singular perturbation, however, regular solutions can be obtained because all points become regular.

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