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Weakly Nonlinear Theory of Rayleigh-Taylor Instability

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Abstract

A weakly nonlinear theory of the Rayleigh-Taylor instability has been developed for initial perturbations with a finite spectrum width. Theoretical result agrees fairly well with hydrodynamic simulation. Saturation of linear growth and spectral broadening in a weakly nonlinear stage are discussed. It is shown that a broad initial bandwidth leads to lower saturation amplitudes of the linear growth compared with that of a single mode case.

Keywords:

Rayleigh-Taylor instability, inertial confinment fusion, weakly nonlinear theory, finite spectrum width

1. Introduction

An interface between two fluids is unstable when the interface is accelerated into the heavier fluid. This instability is known as the Rayleigh-Taylor (RT) instability. Recently RT instability has found a wider spectrum of interest in such fields as geophysics and astrophysics [1] as well as inertial confinement fusion (ICF). The RT instability in ICF target implosion occurs both at the ablation surface in acceleration phase and at the inner shell-fuel interface in stagnation phase. Small perturbation grows to amplitudes so large that the shell breaks up prior to ignition. It is therefore extremely important for ICF to predict the linear and nonlinear growth of the perturbation.

It is generally recognized a single RT unstable mode grows exponentially, until the amplitude is about 1/10 to 1/5 of the wavelength [2,3]. Haan [4] was the first who pointed out importance of a finite spectrum width in nonlinear RT instability. However he a priori assumed saturation amplitude of the linear growth and nonlinear growth rate of individual modes, by using simulation results. Ofer *et al.* [5] extends Haan's modal model by introducing the second order mode coupling, but with the same assumptions as Haan made for the saturation of the linear growth.

In this paper, we have developed, for the first time, a self-consistent nonlinear theory of the RT instability for initial perturbations with a finite spectrum width. The theory takes the third order nonlinearity into account. The third order nonlinearity results in the onset of the linear growth saturation. It will be shown that the saturation amplitude of the linear growth thus obtained agrees well with simulation results. Spectral broadening in a weakly nonlinear stage is also shown. In Sec. 2, governing equations are derived, while results of the governing equations are discussed in Sec. 3.

2. Derivation of Governing Equations

We consider a planar interface between inviscid, incompressible fluids without surface tension. It is not difficult to extend other geometry, such as cylindrical and spherical surfaces, and also to include surface tension. We think of the system in a gravitational field, g, with the interface $z = z(\boldsymbol{x}, t)$ between an upper fluid of density ρ_H and a lower fluid of density ρ_L, where $\boldsymbol{x}$ is a

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two-dimensional vector (x, y), and $\rho_H > \rho_L$. Let $z = 0$ and $z = \xi(\boldsymbol{x}, t)$ be the unperturbed and perturbed surfaces, respectively. Periodic boundary conditions apply in x and y, with box length L.

We consider a perturbation of the surface with a small but finite bandwidth, $\boldsymbol{k} = \boldsymbol{k}_0 + \delta\boldsymbol{k}$, where $\boldsymbol{k}$ has discrete allowed values $(2\pi m/L, 2\pi n/L)$, and $|\boldsymbol{k}_0| > |\delta\boldsymbol{k}|$. Then the perturbation of the interface can be expanded as

$$\xi(\boldsymbol{x}, t) = \sum_{\boldsymbol{k}} \xi_{\boldsymbol{k}}(t)\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = e^{i\boldsymbol{k}_0\cdot\boldsymbol{x}} \sum_{\delta\boldsymbol{k}} \xi_{\boldsymbol{k}_0+\delta\boldsymbol{k}}\, e^{i\delta\boldsymbol{k}\cdot\boldsymbol{x}} \equiv \bar{\xi}(\boldsymbol{x}, t)\, e^{i\boldsymbol{k}_0\cdot\boldsymbol{x}} \tag{1}$$

The amplitude $\bar{\xi}(\boldsymbol{x}, t)$ thus introduced is a slowly varying function in space.

We set up the problem in a standard way [6] for an irrotational flow. Let $\phi_H(\boldsymbol{x}, z, t)$ and $\phi_L(\boldsymbol{x}, z, t)$ be the velocity potentials in the upper and lower fluids respectively. The velocity $\boldsymbol{v}$ is given by $-\nabla\phi_H$ for $z > \xi$ and $-\nabla\phi_L$ for $z > \xi$. From the assumption of incompressibility, the potentials satisfy

$$\nabla^2\phi_H = \nabla^2\phi_L = 0. \tag{2}$$

We require that ϕ_H and ϕ_L go to zero as z goes to $+\infty$ and $-\infty$, respectively. In the same way as the interface perturbation, the velocity potentials are also composed of slowly varying amplitudes as

$$\begin{aligned}\phi_H(\boldsymbol{x}, z, t) &= \bar{\phi}_H(\boldsymbol{x}, z, t)\, e^{i\boldsymbol{k}_0\cdot\boldsymbol{x} - k_0 z},\\ \phi_L(\boldsymbol{x}, z, t) &= \bar{\phi}_L(\boldsymbol{x}, z, t)\, e^{i\boldsymbol{k}_0\cdot\boldsymbol{x} + k_0 z},\end{aligned} \tag{3}$$

where $k_0 = |\boldsymbol{k}_0|$.

From eq.(2), the amplitudes of the velocity potential satisfy

$$\begin{aligned}k_0 \frac{\partial}{\partial z}\bar{\phi}_H &= i\boldsymbol{k}_0\cdot\frac{\partial}{\partial\boldsymbol{x}}\bar{\phi}_H + \frac{1}{2}\left|\frac{\boldsymbol{k}_0}{k_0}\times\nabla\right|^2 \bar{\phi}_H,\\ k_0 \frac{\partial}{\partial z}\bar{\phi}_L &= -i\boldsymbol{k}_0\cdot\frac{\partial}{\partial\boldsymbol{x}}\bar{\phi}_L - \frac{1}{2}\left|\frac{\boldsymbol{k}_0}{k_0}\times\nabla\right|^2 \bar{\phi}_L,\end{aligned} \tag{4}$$

where $\nabla \equiv \partial/\partial\boldsymbol{x}$ and we have neglected the third order spatial derivatives of the amplitudes with respcet to $\boldsymbol{x}$.

From the definition of the surface perturbation $\xi(\boldsymbol{x}, t)$,

$$\frac{d\xi}{dt} = \frac{\partial\xi}{\partial t} - \frac{\partial\xi}{\partial x}\frac{\partial\phi}{\partial x}\bigg|_\xi - \frac{\partial\xi}{\partial y}\frac{\partial\phi}{\partial y}\bigg|_\xi = -\frac{\partial\phi}{\partial z}\bigg|_\xi, \tag{5}$$

where ϕ is either ϕ_H or ϕ_L. Since the normal component of the fluid velocity is continuous across the interface, $\partial\xi/\partial t$ evaluated with either ϕ_H of ϕ_L should coincide with each other, namely

$$\begin{aligned}\frac{\partial\xi}{\partial t} &= \frac{\partial\xi}{\partial x}\frac{\partial\phi_H}{\partial x}\bigg|_\xi + \frac{\partial\xi}{\partial y}\frac{\partial\phi_H}{\partial y}\bigg|_\xi - \frac{\partial\phi_H}{\partial z}\bigg|_\xi\\ &= \frac{\partial\xi}{\partial x}\frac{\partial\phi_L}{\partial x}\bigg|_\xi + \frac{\partial\xi}{\partial y}\frac{\partial\phi_L}{\partial y}\bigg|_\xi - \frac{\partial\phi_L}{\partial z}\bigg|_\xi.\end{aligned} \tag{6}$$

From equation of motion, the pressure in each fluid is given by Bernoulli's equation,

$$p = \rho\left[\frac{\partial\phi}{\partial t} - \frac{1}{2}(\nabla\phi)^2 - g\xi\right]. \tag{7}$$

At the interface the pressure should be balanced,

$$p_H\Big|_\xi = p_L\Big|_\xi, \tag{8}$$

where p_H and p_L are evaluated with ϕ_H and ρ_H or ϕ_L and ρ_L in eq.(7), respectively. Substituting eq.(7) into eq.(8), we obtain

$$\rho_H\left[\frac{\partial\phi_H}{\partial t}\bigg|_\xi - \frac{1}{2}(\nabla\phi_H)^2\Big|_\xi - g\xi\right] = \rho_L\left[\frac{\partial\phi_L}{\partial t}\bigg|_\xi - \frac{1}{2}(\nabla\phi_L)^2\Big|_\xi - g\xi\right]. \tag{9}$$

In eqs.(6) and (9), the partial derivative of the velocity potential at the interface can be calculated by Taylor series expansion with respect to the perturbation amplitude under the assumption of $k_0|\bar{\xi}| \sim o(\varepsilon) < 1$. Nonlinear terms in eqs.(6) and (9) generate higher harmonics. Therefore the interface perturbation and velocity potential can be written as

$$\begin{aligned}\xi(\boldsymbol{x}, t) &= \sum_{n=0}^{\infty} \bar{\xi}^{(n)}(\boldsymbol{x}, t)\, e^{in\boldsymbol{k}_0\cdot\boldsymbol{x}},\\ \phi_H(\boldsymbol{x}, t) &= \sum_{n=0}^{\infty} \bar{\phi}_H^{(n)}(\boldsymbol{x}, z, t)\, e^{in\boldsymbol{k}_0\cdot\boldsymbol{x} - nk_0 z},\\ \phi_L(\boldsymbol{x}, t) &= \sum_{n=0}^{\infty} \bar{\phi}_L^{(n)}(\boldsymbol{x}, z, t)\, e^{in\boldsymbol{k}_0\cdot\boldsymbol{x} + nk_0 z}.\end{aligned} \tag{10}$$

We consider up to the third order nonlinearity in order to describe the self modulation of the perturbation. In eq.(10), $n = 0$ and $n = 2$ terms arise form the mode coupling between k_0 modes and they are in the second order of ε^2, while $n = 3$ term arises from the mode coupling between k_0 mode and $2k_0$ mode as the third order of ε^3. It should be noted that $n = 1$ term contains the third order nonlinearity due to the mode coupling, for example, between $2k_0$ mode and k_0 mode ($2k_0 - k_0 = k_0$). We also assume that the bandwidth is of the order of ε, i.e. $\delta k/k_0 \sim o(\varepsilon)$, and $\partial\bar{\xi}/\partial x \sim o(\varepsilon^2)$. After tedious calculation, we obtain the governing equations as

$$\frac{\partial^2 \phi^{(1)}}{\partial t^2} = \gamma_1^2 \phi^{(1)} - \mathrm{i}\frac{\gamma_1^2}{k_0}\left(\frac{\boldsymbol{k}_0}{k_0}\cdot\nabla\right)\phi^{(1)} - \frac{\gamma_1^2}{2k_0^2}\left|\frac{\boldsymbol{k}_0}{k_0}\times\nabla\right|^2\phi^{(1)} + \frac{\alpha k_0^2(2\gamma_1^2+3\gamma_2^2)}{2\gamma_1^2}\frac{\partial\phi^{(1)*}}{\partial t}\cdot\phi^{(2)}$$
$$+\frac{k_0^2(4\gamma_1^2\rho_H - \gamma_2^2\rho_L)}{\gamma_1^2(\rho_H+\rho_L)}\phi^{(1)*}\cdot\frac{\partial\phi^{(2)}}{\partial t} - \frac{k_0^4\alpha(\rho_H+2\rho_L)}{\gamma_1^2(\rho_H+\rho_L)}\left|\frac{\partial\phi^{(1)}}{\partial t}\right|^2\phi^{(1)}$$
$$-\frac{k_0^4(\rho_H-2\rho_L)}{\gamma_1^2(\rho_H+\rho_L)}\phi^{(1)*}\cdot\left(\frac{\partial\phi^{(1)}}{\partial t}\right)^2 - \frac{k_0^4(\rho_H^2-3\rho_L^2)}{(\rho_H+\rho_L)^2}\left|\phi^{(1)}\right|^2\phi^{(1)}, \qquad (11a)$$

where $\phi^{(2)}$ is given by the second order equation,

$$\frac{\partial^2 \phi^{(2)}}{\partial t^2} = \gamma_2^2\phi^{(2)} + \frac{2k_0^2\rho_L}{\rho_H+\rho_L}\phi^{(1)}\frac{\partial\phi^{(1)}}{\partial t}. \qquad (11b)$$

The surface perturbation amplitude is written as a function of the velocity potential as,

$$\xi^{(1)} = \frac{k_0}{\gamma_1^2}\frac{\partial\phi^{(1)}}{\partial t} - \frac{k_0^3(3\rho_H-\rho_L)}{2\gamma_1^4(\rho_H+\rho_L)}\frac{\partial\phi^{(1)*}}{\partial t}\cdot\frac{\partial\phi^{(2)}}{\partial t} - \frac{k_0^3}{\gamma_1^2(\rho_H+\rho_L)}\left(2\rho_H - \frac{\gamma_2^2}{\gamma_1^2}\rho_L\right)\phi^{(1)*}\cdot\phi^{(2)}$$
$$+\frac{k_0^5(5\rho_H-3\rho_L)}{8\gamma_1^6(\rho_H+\rho_L)}\left|\frac{\partial\phi^{(1)}}{\partial t}\right|^2\frac{\partial\phi^{(1)}}{\partial t} + \frac{k_0^5\rho_L(\rho_H-3\rho_L)}{2\gamma_1^4(\rho_H+\rho_L)^2}\frac{\partial\phi^{(1)*}}{\partial t}\cdot\phi^{(1)2} + \frac{k_0^5\alpha}{\gamma_1^4}\left|\phi^{(1)}\right|^2\frac{\partial\phi^{(1)}}{\partial t}, \qquad (12)$$

where we omit the bar for the slowly varying amplitudes and the subscript H for ϕ. $\gamma_1 = \sqrt{\alpha k_0 g}$ and $\gamma_2 = \sqrt{2\alpha k_0 g}$ are the linear growth rates corresponding to the wavenumber k_0 and that of the second harmonics, respectively, and $\alpha = (\rho_H - \rho_L)/(\rho_H + \rho_L)$ is the Atwood number.

3. Results and Discussion

First, we show the validity of the governing equations by comparing with two dimensional hydrodynamic simulation. To solve eqs.(11a) and (11b) numerically, we define the slowly varying velocity potential at grid points of which number and grid size are chosen to be $\bar{L} = 32$ and $k_0/\Delta x = 3\pi$, respectively. Therefore the minimum wave number becomes $\delta k_{\min}/k_0 = 2\pi/k_0 L\Delta x = 1/48$. We choose a Gaussian distribution function with the bandwidth of $\delta k/k_0 = 10^{-2}$ as an initial spectrum of the velocity perturbation of which root mean square(rms) equals to $\sqrt{<(k_0\dot{\xi}/\gamma_1)^2>} = 10^{-3}$, and $\xi = 0$. Two dimensional simulation is carried out with the use of a hydrodynamic simulation code, IMPACT-2D [7]. In the analytical model, we consider only the slowly varying amplitude of the velocity potential, while in the hydrodynamic simulation we must use fine meshes to resolve perturbations with wavelength of the order of $2\pi/k_0$. We used mesh numbers of $L_x L_z = 2048 \times 300$. The wavelength corresponding to k_0 is chosen to be $\lambda = L_x/m$ and $m = 48$, so that the difference of the wave number between the modes with m and $m \pm 1$ coincides with the minimum wave number $\delta k_{\min}$ in the analytical model. Initial velocity perturbations are given for mode numbers within $m \pm \bar{L}/2 = 48 \pm 16$, with the same amplitudes as those of the governing equations. The Atwood number is chosen to be $\alpha = 0.82$.

Figure 1(a) shows the rms velocity amplitudes as a function of time, where solid line is solution of eq.(11), and circles are simulation results. Saturation of linear growth occurs around time of $\gamma_1 t = 6\sim7$. Normalized velocity amplitudes of each mode from 45 to 51 are shown in Fig. 1(b), simulation results, and Fig. 1(c), solutions of eq.(11), respectively. Dips of the veloocity perturbations appeared in Figs. 1(b) and (c) correspond to phase change due to nonlinear interaction. As shown in Figs. 1(a), (b) and (c), solutions of eq.(11) agree fairly well with simulation results.

We now consider three dimensional instability, by choosing $\bar{L}^2 = (512)^2$, $k_0\Delta x = 3\pi$ and $\alpha = 0.8$. The minimum wavenumber then becomes $\delta k_{\min}/k_0 = 1/768$. The initial spectrum of the interface displacement ($\xi(t = 0) \neq 0$, $\dot{\xi} = 0$) is assumed to be a Gaussian distribution function with the bandwidth of $\delta k/k_0 = 10^{-2}$. Figure 2 shows the time evolution of rms amplitude, and individual mode amplitudes (0, 0), (±10, ±10), (±20, ±20), and (±30, ±30). Exponential growth of rms amplitude starts to saturate when it reaches approximately to $\sqrt{<(k_0\xi)^2>} \sim 0.03$ around time $\gamma_1 t \sim 4$. This saturation amplitude of the linear growth is much less than that of a single mode case. The broader initial bandwidth leads to the lower saturation rms amplitude. As shown in Fig. 2, the individual modes, $(k_{0x} + m\Delta k_{\min}, k_{0y} + n\Delta k_{\min})$ within $|m| \leq 10$, $|n| \leq 10$ also starts to saturate at much lower amplitudes. The details of the saturation of linear growth will be discussed elsewhere.

In the weakly nonlinear stage, the large amplitude modes ($|m| \leq 5$, $|n| \leq 5$) grow linearly in time, while much higher modes ($|m| \geq 20$, $|n| \geq 20$) grows very rapidly due to nonlinear interaction as shown in Fig. 2. Until time $\gamma_1 t = 6$, the rms amplitude grows much slowly compared with the linear growth. After that time, many modes become comparable to each other. Around time $\gamma_1 t = 8$, the normalized rms amplitudes become

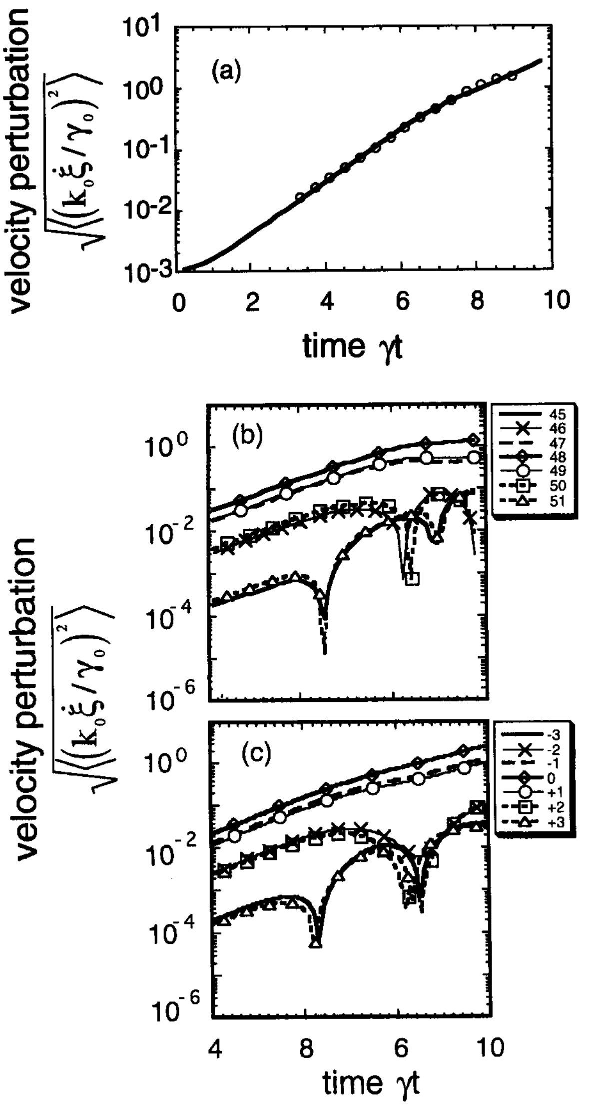

Fig. 1 (a) Root mean square of velocity perturbations normalized by k_0/γ_1, as functions of normalized time $\gamma_1 t$; solid line is analytical result, while circles are simulation results. Normalized velocity perturbations of each mode from modes 45 to 51, (b) simulation results and (c) analytical results.

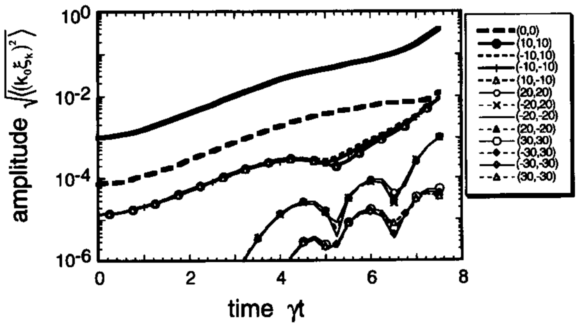

Fig. 2 Normalized amplitudes of displacement $k_0\xi$, as functions of normalized time $\gamma_1 t$. Solid line is root mean square amplitude, and others lines represent amplitudes of each $k_0\xi$ mode, $(k_{0x} + m\delta k_{\min}, k_{0y} + n\delta k_{\min})$, (m, n) are indicated in the figure.

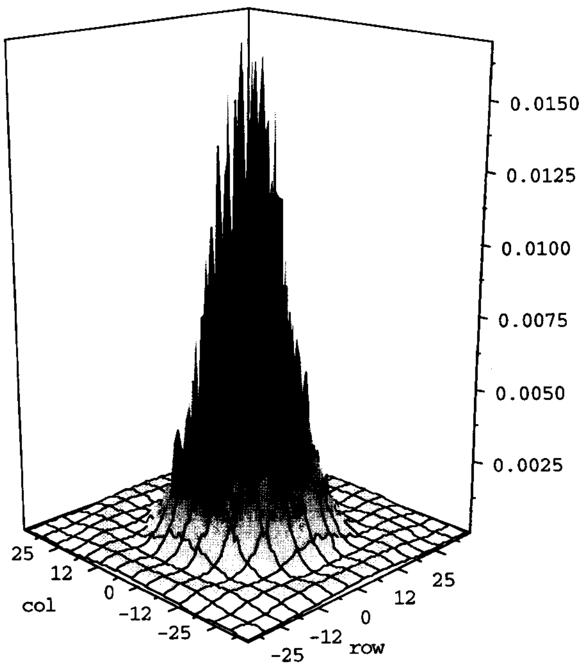

Fig. 3 Spectrum in weakly nonlinear stage, $k_0\xi(m, n)$, integers in row and col indicates $(m, n) = (k_{0x} + m\delta k_{\min}, k_{0y} + n\delta k_{\min})$.

greater than one, and the weak nonlinear theory breaks. However it is found that if $\gamma_2 = 0$, the weakly nonlinear stage continues much longer time, namely $\sqrt{<(k_0\xi)^2>} \leq 1$ up to $\gamma_1 t \sim 13$. The condition of $\gamma_2 = 0$ can be realized by the mass ablation effect [8] at the ablation surface in ICF or the surface tension. The long duration of the weakly nonlinear stage with $\gamma_2 = 0$ predicts the reduction of the nonlinear RT growth at the ablation surface in ICF, not only in the linear growth rate.

In the weakly nonlinear stage, the broadening of the band width occurs as expected. The bandwidth becomes 1.5 times greater than its initial value at time $\gamma_1 t = 6$. The spectrum of the interface displacement at $\gamma_1 t = 7$ is shown in Fig. 3.

4. Conclusions

We have developed a weakly nonlinear theory of the Rayleigh-Taylor instability with a finite spectrum width. In the modal model [4,5], they a priori assume that the linear growth starts to saturate when the root mean square amplitudes reach values comparable to or slightly greater than the saturation amplitude of a single mode. The self-consistent theory developed includes the third order nonlinearity, and thus the theory predicts the onset of the linear growth saturation for perturbations with a finite spectrum width without any assumptions.

Theoretical result agrees fairly well with two dimensional hydrodynamic simulation, including nonlinear phase change of individual modes. It is also shown that a broad initial bandwidth leads to low saturation root mean square amplitude of the linear growth. This is quite different from that previously expected in the modal model.

In a weakly nonlinear regime, large amplitude modes grow linearly in time in the same way as a single mode case. However many modes are excited very rapidly due to nonlinear interaction, and broadening of the bandwidth occurs. The mass ablation may reduce not only linear growth but also weakly nonlinear growth although further investigation is required.

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