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Spheromak as a Non-Taylor Relaxed State

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Abstract

The Spheromak Configuration, which is generally believed to be a minimum energy, force-free and zero pressure gradient state, is shown to evolve as a relaxed state with minimum dissipation. The Euler-Lagrange equation for the minimum dissipation relaxed state can be solved in terms of Chandrasekhar Kendall eigenfunctions analytically continued in the complex domain. This state is not force-free and has non zero pressure gradient and further shows the non constancy of the ratio of parallel current to the magnetic field. This is consistent with many experimental measuremets on Spheromak.

Keywords:
relaxation, spheromak, self-organization, minimum dissipation

1. Introduction

The Spheromak is a magnetic confinement system, where the magnetic field is self-organized to form a set of closed nested surfaces in a region of space. The Spheromak configuration corresponds to the classical "Hill's Vortex" of fluid dynamics. The toroidal and poloidal fields of a Spheromak are of equal strength approximately and are generated primarily by internal plasma currents rather than external coils.

Important theoretical and experimental results for the spheromak problem came from the application of Taylor's relaxation model [1] that conjectured the magnetic fields in a plasma to relax towards a state of minimum energy subject to the constraint of constant magnetic helicity. In a closed system, the minimum magnetic energy equilibrium satisfies the force free equation $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$ with λ = constant. Rosenbluth and Busaac [2] obtained the spheromak field configuration from the Chandrasekhar Kendal (CK) solution of the force-free equation and thus attempted to confirm that spheromak is a Taylor force-free state.

In reality, any plasma confining device must have a non-flat pressure profile. Also, since competing effects are present in any experiment, deviations from a uniform, constant λ can be expected. For spheromak, such pressure profiles have been observed experimentally and departure from minimum energy state was also noticed [3-6].

Montgomery & Phillips [7] utilized the principle of minimum dissipation rate for the first time in an MHD problem to predict the relaxed states of a RFP configuration. We try to establish here that a spheromak equilibrium is a non force-free relaxed state, and supports a significant fraction of perpendicular component of current as well as a pressure gradient by modelling the equilibrium as a state of minimum dissipation rate with constant helicity.

2. Euler Lagrange Equation

The ohmic dissipation rate [7] for a magnetofluid is given by

$$R = \eta \int j^2 \mathrm{d}\tau, \tag{1}$$

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where η is the plasma resistivity and the integral is over the entire confinement region. The magnetic helicity $K = \int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}\tau$ is an invariant of motion in ideal magnetihydrodynamics. If the turbulence is sufficiently low, K still serves as a constraint as it decays at a slower rate compared to R. If the energy dissipation rate given by eq. (1) is minimized by including helicity as a constraint on the minimization through the use of Lagrange's multiplier $\bar{\lambda}$, The following variational equation is obtained

$$\delta \int (\eta j^2 + \bar{\lambda}\boldsymbol{A}\cdot\boldsymbol{B})\,\mathrm{d}\tau, \qquad (2)$$

Since this must hold for arbitrary $\mathrm{d}\tau$, we obtain the Euler-Lagrange equation :

$$\nabla\times\nabla\times\nabla\times\boldsymbol{B} = \Lambda\boldsymbol{B}, \qquad (3)$$

where $\Lambda = \bar{\lambda}/\eta$ is a constant.

3. Spheromak Solutions of $\nabla \times \nabla \times \nabla \times \boldsymbol{B} = \Lambda\boldsymbol{B}$

A solution of eq. (3) is constructed as a linear combination of the solutions of the force free equation $\nabla \times \boldsymbol{B}_0 = \lambda\boldsymbol{B}_0$ which are given in terms of Chandrasekhar-Kendall eigenfunctions. The magnetic field $\boldsymbol{B}_0$, being a solenoidal field can be decomposed into its toroidal and poloidal ingredients:

$$\boldsymbol{B}_0 = \boldsymbol{B}_{0\mathrm{T}} + \boldsymbol{B}_{0\mathrm{P}},$$
$$\boldsymbol{B}_{0\mathrm{T}} = \nabla\times(\xi\Phi_0),\ \boldsymbol{B}_{0\mathrm{P}} = \nabla\times\nabla\times(\xi\Psi_0). \qquad (4)$$

Here ξ is a position vector, and $\boldsymbol{B}_0$ satisfies the Taylor equation provided

$$(\nabla^2+\lambda^2)\,\Phi_0 = 0,\ \ \Psi_0 = \Phi_0/\lambda.$$

In spherical polar coordinates (r, θ, ϕ)

$$\zeta = r,\ \ \Psi_0(r,\theta) = \frac{\alpha_0}{r}\,\mathrm{j}_m(\lambda r)\mathrm{P}_m^n(\cos\theta)\,e^{in\phi}, \qquad (5)$$

Here, $\mathrm{j}_m(\lambda r)$ is a spherical Bessel function, $\mathrm{P}_m^n(\cos\theta)$ is an Associated Legendre function.

The classical spheromak equilibrium solution is given by $n = 0$, $m = 1$ state and the corresponding flux function χ_0 that describes the lines of constant poloidal magnetic field is obtained as $\chi_0 = -r\sin\theta\,\mathrm{d}\Psi_0/\mathrm{d}\theta$.

The solutions of eq. (3) are obtained as the analytic continuation of the solutions of $\nabla \times \boldsymbol{B}_0 = \lambda\boldsymbol{B}_0$ to complex values of λ. Expressing $\boldsymbol{B}$ as in eq. (4)

$$\boldsymbol{B} = \nabla\times(\vec{\xi}\Phi) + \nabla\times\nabla\times(\vec{\xi}\Psi), \qquad (6)$$

with

$$\Phi = \sum_{i=0}^{2}\alpha_i\Phi_i,\ \ \Psi = \frac{1}{\lambda}\sum_{i=0}^{2}\alpha_i\omega^{2i}\Phi_i, \qquad (7)$$

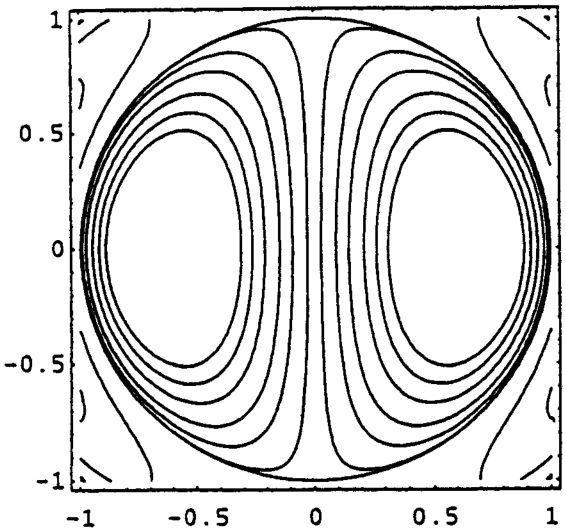

Fig. 1 The Spheromak

where Φ_i are the solutions of $(\nabla^2 + \lambda^2\omega^{2i})\Phi_i = 0$ given as

$$\Phi_i = \mathrm{j}_m(\lambda\omega^i r)\mathrm{P}_m^n(\cos\theta)\,e^{in\phi}. \qquad (8)$$

The expression for $\boldsymbol{B}$ given by eq. (6) then satisfies eq. (3) for $\Lambda = \lambda^3$. The corresponding flux function χ is given by

$$\chi(r,\theta) = \frac{r}{\lambda}\sum_{i=0}^{2}\alpha_i\omega^{2i}\mathrm{j}_1(\lambda\omega^i r)\sin^2\theta.$$

In eqs. (7), the constants α_i are fixed by assuming the conducting boundary conditions

$$\boldsymbol{B}\cdot\boldsymbol{n} = 0,\ \ \boldsymbol{J}\times\boldsymbol{n} = 0\ \ at\ \ r = a, \qquad (9)$$

where a is the radius of the sphere. For a given vaue of n and m, the value of λa can be obtained from the boundary conditions. The eigensolution with lowest energy dissipation rate for the $n = 0$, $m = 1$ state has the eigenvalue $\lambda a = 3.99$. The $\boldsymbol{B}_\mathrm{P}$ lines, given by χ = constant are sketched in Fig. 1 for the lowest energy dissipation state.

4. Results

For values of $\lambda a = 3.99$, which corresponds to the state with lowest energy dissipation rate, the plots of $j_\parallel/B$ and $j_\perp/B$ in the $z = 0$ plane are shown in Fig. 2. A non constant $j_\parallel/B$ profile is observed which is in contrast to the usual Taylor state for which λ = constant. Also a significant amount of perpendicular component of current is obtained. This is similar to some of the experimental observations [2-5], where a non constant $j_\parallel/B$ profile is obtained. The profiles of toroidal and

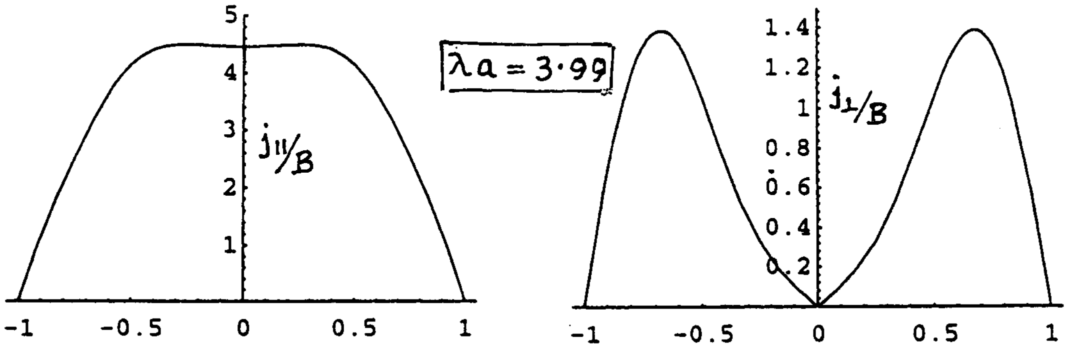

Fig. 2 A plot of $j_{\parallel}/B$ and $j_{\perp}/B$ against r/a in the $z = 0$ plane for $\lambda a = 3.99$.

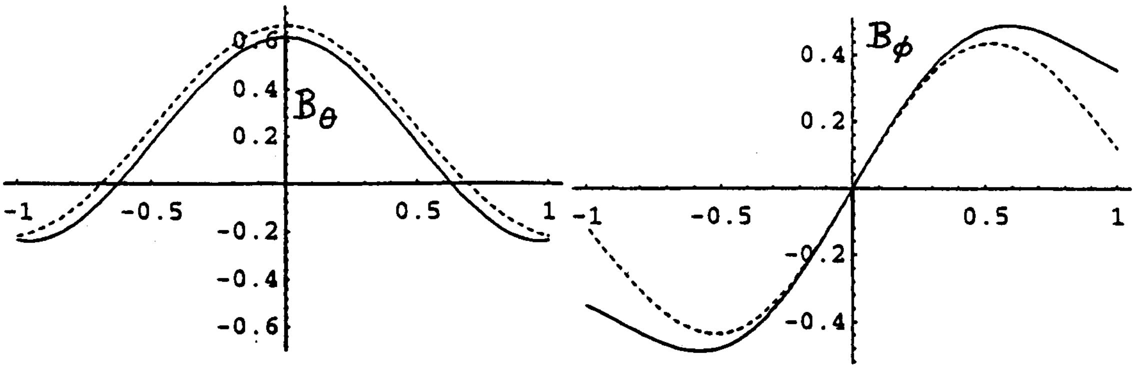

Fig. 3 The profiles of B_θ and B_ϕ against r/a in the $z = 0$ plane for the state $\lambda a = 3.99$. The dotted curves show the field profiles corresponding to the Taylor state for the same eigenvalue.

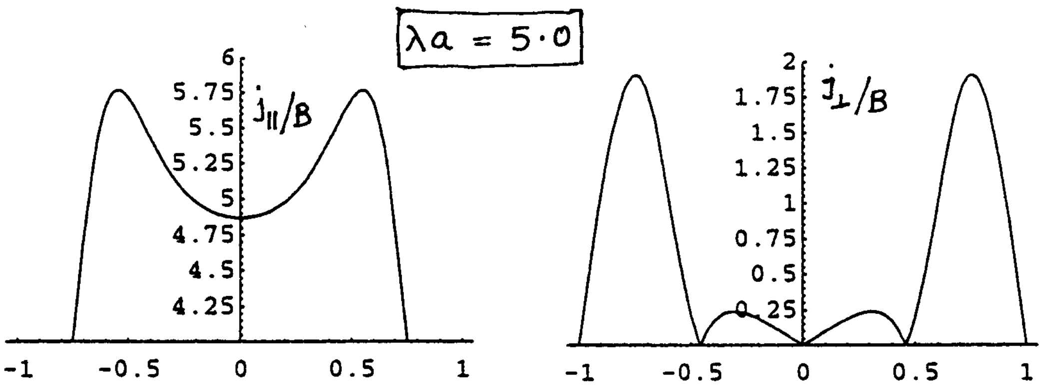

Fig. 4 A plot of $j_{\parallel}/B$ and $j_{\perp}/B$ against r/a in the $z = 0$ plane for $\lambda a = 5.0$.

poloidal magnetic field components in the $z = 0$ plane are shown in Fig. 3. The profiles are similar to those observed in ref. [3].

For $\lambda a = 5$ which is higher than the value corresponding to the lowest energy dissipation state, the profiles of $j_{\parallel}/B$ and $j_{\perp}/B$ are shown in Fig. 4. For this value of λa, the fields, however, do not satisfy the boundary conditions. In this case, a double humped profile for the ratio $j_{\parallel}/B$. is sketched in Fig. 4. This state also supports a perpendicular component of current

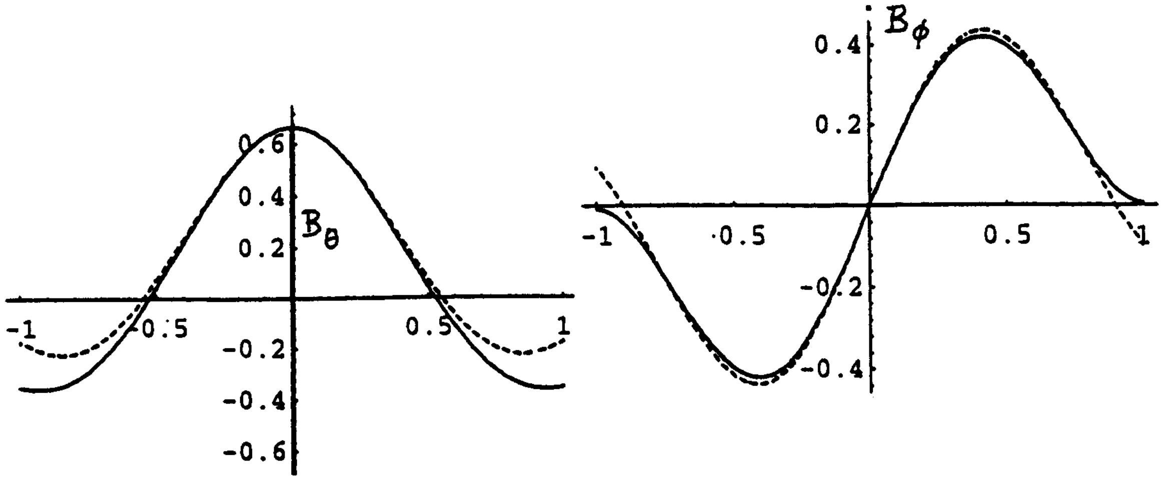

Fig. 5 The profiles of B_θ and B_ϕ against r/a in the $z = 0$ plane for $\lambda a = 5.0$.

showing departure from the usual Taylor state. A similar observation for $j_{\|}/B$ was also reported by Hart et. al. [3]. Also, such double humped profile for $j_{\|}/B$ was obtained by solving numerically the Grad-Shafranov equation [3,8] for a spheromak with an oblate conducting boundary. The profiles of toroidal and poloidal magnetic field components in the $z = 0$ plane are shown in Fig. 5.

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