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Relaxation of a Magnetized Plasma to States Other than Force-free

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Abstract

A theory of relaxation of a magnetized plasma based on the principle of minimum dissipation and constancy of global magnetic helicity is presented. The theory explains the occurence of a wide variety of relaxed states, which are not necessarily force free, in simulations and laboratory experiments. A solution of the Euler Lagrange equation describing the states of minimum dissipation is accomplished using the analytic continuation of the Chandrasekhar-Kendall eigenfunctions in the complex domain. Explicit forms of the solutions are constructed using appropriate boundary conditions at the boundary. The solutions, as expected, support a finite pressure gradient. The distinct feature of this theory is to show that it is possible to realize MHD equilibria with finite pressure gradient in a single fluid system even without a long-term coupling between mechanical flow and magnetic field.

Keywords:

relaxation, reversed-field pinches, self-organization, minimum dissipation

1. Introduction

The relaxed states of a magnetoplasma, according to Taylor's conjecture [1], are obtained by minimizing the total magnetic energy under the constraint of total magnetic helicity defined by $K = \int \boldsymbol{A}\cdot\boldsymbol{B}\mathrm{d}V$. A variational technique leads to the Euler-Lagrange's equation $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$, where λ is a constant. The relaxed state is a force-free state.

Taylor's theory, although quite successful in explaining a number of experimental results, including those of RFP, is viewed as inadequate by many workers. Relaxed states as envisaged by Taylor have only zero pressure gradient. Extensive numerical works by Sato and his collaborators have established [2] the existence of self-organized states with finite pressure, i.e. these states are governed by the magneto-hydrodynamic force balance relation $\boldsymbol{J} \times \boldsymbol{B} = \nabla p$. Several attempts [3,4] have been made in the past to obtain relaxed states which could support finite pressure gradient, a large number of them making use of the coupling of the flow with magnetic field. The novel feature of our work is to show that it is possible for a single fluid to relax to an MHD equilibrium with a magnetic field configuration which can support pressure gradient, even without a long-term coupling between the flow and the magnetic field.

The concept of minimum dissipation rate was used for the first time by Montgomery and Phillips [5] in an MHD problem to understand the steady state profiles of RFP configuration under the constraint of constant rate

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of supply and dissipation of helicity and the usual physical boundary conditions for a conducting wall. It may be pointed out that the principle was also discussed by Chandrasekhar and Woltzer [6] in a sequel to the complete general solution of the force-free equation by Chandrasekhar and Kendall [7]. It is also our conjecture that relaxed states could be characterized as the states of minimum dissipation rather than states of minimum energy.

2. Euler-Lagrange Equation

We consider a closed system of an incompressible, resistive magnetofluid, without any mean flow velocity, described by the standard MHD equations in presence of a small but finite resistivity η. In presence of small scale turbulence in the system, we may expect that the rate of energy dissipation decays at a faster rate than helicity. We therefore minimize the ohmic dissipation $R = \int \eta j^2 \mathrm{d}V$ subject to the constraints of helicity $\int \boldsymbol{A}\cdot\boldsymbol{B}\mathrm{d}V$. A variational principle leads to the Euler-Lagrange equation

$$\nabla\times\nabla\times\nabla\times\boldsymbol{B} = \Lambda\boldsymbol{B}, \tag{1}$$

where, Λ is a undetermined multiplier.

3. Solutions of $\nabla\times\nabla\times\nabla\times\boldsymbol{B} = \Lambda\boldsymbol{B}$

A solution of eq. (1) can be obtained in terms of the Chandrasekhar Kendall eigenfunctions [7] which are solutions of the equation $\nabla\times\boldsymbol{B}_0 = \lambda\boldsymbol{B}_0$. The solution of eq. (1) can be written as

$$\boldsymbol{B} = \sum_{n=0}^{2} \alpha_n \boldsymbol{B}_n , \tag{2}$$

Here α_n are constants and $\boldsymbol{B}_n$ are solutions of the force-free equation for complex λ such that $\nabla\times\boldsymbol{B}_n = \lambda\omega^n \boldsymbol{B}_n$.

$$\boldsymbol{B}_n = \lambda\omega^n\nabla\Phi_n\times\nabla z + \nabla\times(\nabla\Phi_n\times\nabla z) \quad n = 0,1,2\ . \tag{3}$$

with $\Phi_n = J_m(\mu_n r)\exp[\mathrm{i}(m\theta + kz)]$, $\mu_n^2 + k^2 = \lambda^2\omega^{2n}$, $\omega = \exp(2\pi\mathrm{i}/3)$. It can be easily demonstrated that the expression for $\boldsymbol{B}$ given in eq. (2) satisfies eq. (1) for $\Lambda = \lambda^3$.

The constants α_2 and α_1 as well as the values of λa are fixed by assuming the boundary conditions for a perfectly conducting boundary wall, given by

$$\boldsymbol{B}\cdot\boldsymbol{n} = 0,\ \ \boldsymbol{j}\times\boldsymbol{n} = 0 \quad at \ \ r = a.$$

It is to be noted that for the cylindrically symmetric ($m = 0$, $k = 0$) state the boundary condition is trivially satisfied and hence does not determine λa. To get the numerical value of λ for $m \neq 0$, we utilize the boundary conditions and obtain $\lambda a = 3.11$ and $ka = 1.23$ as the minimum values of λa and ka for the $m = 1$ state.

The magnetic field components for the $m = 0$, $k = 0$ state are given by

$$B_r = 0\ ,$$

$$B_\theta = \lambda^2\alpha_0\left[\mathrm{J}_1(\lambda r) + 2\mathrm{Re}\left(\frac{\alpha_1}{\alpha_0}\omega^2\,\mathrm{J}_1(\lambda\omega r)\right)\right],$$

$$B_z = \lambda^2\alpha_0\left[\mathrm{J}_0(\lambda r) + 2\mathrm{Re}\left(\frac{\alpha_2}{\alpha_1}\omega^2\,\mathrm{J}_0(\lambda\omega r)\right)\right]. \tag{4}$$

4. Toroidal Flux, Field Reversal and Pinch Parameters

Several quantities that have proven useful in describing the laboratory experiments are the toroidal flux, Φ_z field reversal parameter F, pinch parameter Θ and the helicity integral K.

$$\Phi_z = 2\pi\alpha_0\,\lambda a\left[\mathrm{J}_1(\lambda a) + 2\mathrm{Re}\left[\frac{\alpha_1}{\alpha_0}\,\omega\mathrm{J}_1(\lambda\omega a)\right]\right],$$

$$F = \frac{B_z(a)}{<B_z>} = \frac{\lambda a}{2}\,\frac{\mathrm{J}_0(\lambda a) + (\alpha_1/\alpha_0)\omega^2\mathrm{J}_0(\lambda\omega a) + (\alpha_2/\alpha_0)\omega\mathrm{J}_0(\lambda\omega^2 a)}{\mathrm{J}_1(\lambda a) + (\alpha_1/\alpha_0)\omega\mathrm{J}_1(\lambda\omega a) + (\alpha_2/\alpha_0)\omega^2\mathrm{J}_1(\lambda\omega^2 a)},$$

$$\Theta = \frac{B_\theta(a)}{<B_z>} = \frac{\lambda a}{2}\,\frac{\mathrm{J}_1(\lambda a) + (\alpha_1/\alpha_0)\omega^2\mathrm{J}_1(\lambda\omega a) + (\alpha_2/\alpha_0)\omega\mathrm{J}_1(\lambda\omega^2 a)}{\mathrm{J}_1(\lambda a) + (\alpha_1/\alpha_0)\omega\mathrm{J}_1(\lambda\omega a) + (\alpha_2/\alpha_0)\omega^2\mathrm{J}_1(\lambda\omega^2 a)},$$

The toroidal flux parameter is obtained from $\Phi_z = 2\pi\int B_z\, r\mathrm{d}r$. Also, $<B_z>$ refers to the volume average of B_z. In cylindrical coordinates the helicity integral is defined by

$$K = 4\pi^2 R\int_0^a (A_\theta B_\theta + A_z B_z) r\mathrm{d}r\ ,$$

where $2\pi R$ and a are the length and radius of the cylinder. For the minimum value of $\lambda a = 3.11$, corresponding to the cylindrically symmetric state $m = 0$, $k = 0$

$$K/\,\Phi_z^2 = 12.8R/a\ ,$$

where the unit of K/Φ_z^2 is volt-sec. For values of volt-sec 12.8 R/a, $m = 1$, $\lambda a = 3.11$ is the minimum dissipation state.

The pressure profile can be obtained from the relation $\boldsymbol{j} \times \boldsymbol{B} = \nabla p$. For the $m = 0$, $k = 0$ state, the only nonvanishing component of the pressure gradient exists in the radial direction, B_r being zero. The radial pressure distribution is obtained from

$$p(r) = \int (j_\theta B_z - j_z B_\theta)\,\mathrm{d}r\,.$$

5. Results

The toroidal magnetic field profile, B_z is plotted against r/a in Fig. 1 and shows a reversal near the edge of the plasma. The profile of the toroidal current J_z is also shown plotted in Fig. 2. The current vanishes at the boundary because of the boundary conditions we have chosen.

The values of both F and Θ at the boundary $r = a$ are evaluated and F is shown plotted against pinch ratio Θ in Fig. 3. It is observed that F reverses at a value of $\Theta = 2.4$, ($\lambda a = 2.95$) whereas for the Taylor state the reversal is achieved at $\Theta = 1.2$. However, this field reversed state supports pressure gradient inconstrast to the Taylor state.

The pressure profile is shown in Fig. 4 for the $m = k = 0$ state with $\lambda a = 3.0$ which is the minimum energy

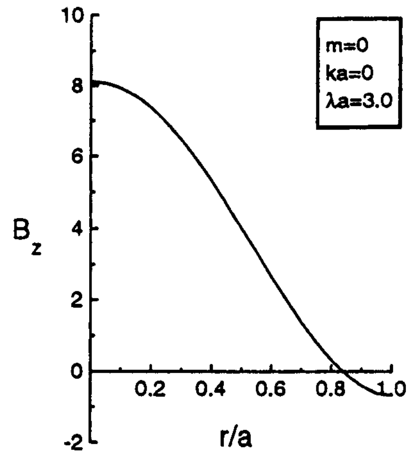

Fig. 1 A plot of the B_z against r/a for the cylindrically symmetric state.

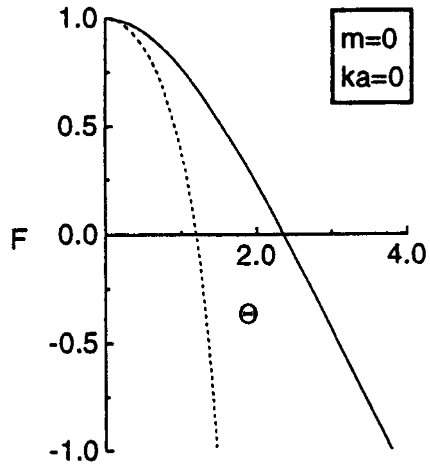

Fig. 3 Field reversal parameter F is shown plotted against the pinch parameter Θ. The dotted curve corresponds to the Taylor state.

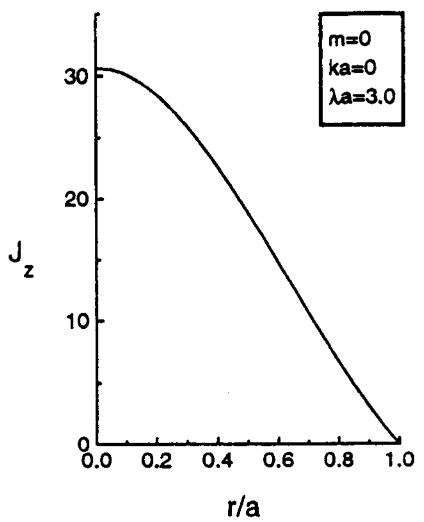

Fig. 2 A plot of the j_z against r/a for the cylindrically symmetric state.

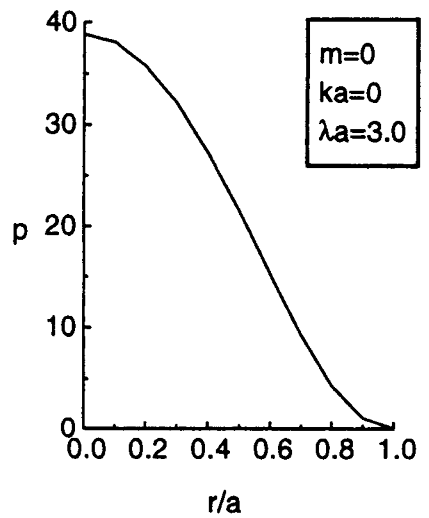

Fig. 4 The radial pressure profile vs r/a.

dissipation, field reversed state.

To conclude, the principle of minimum dissipation is utilized together with the constraints of constant magnetic helicity to determine the relaxed states of a magnetoplasma not driven externally. This relaxed state obtained from single fluid MHD supports pressure gradient. This establishes that a coupling between magnetic field and flow is not an essential criterion for having a non-zero pressure gradient. Further, it is shown that a non force-free state with field reversal properties can exist.

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