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Statistical Theory of Passive Scalar Advected by Turbulence: An Application of the Lagrangian Direct-Interaction Approximation

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Abstract

A closure theory of turbulence so-called the Lagrangian direct-interaction approximation is formulated for a passive scalar field advected by isotropic turbulence. Solutions to the resultant closure equations for the correlation functions of scalar and velocity fields are shown to be completely consistent with well-known phenomenologies on small-scale statistics of turbulence.

Keyword:
statistical theory of turbulence, direct-interaction approximation, passive scalar field

1. Introduction

We investigate a passive scalar field $\theta(\boldsymbol{x}, t)$, e.g., temperature, particle concentration, dye, smoke, etc, which is advected by turbulent velocity field and diffused by molecular diffusion as

$$\frac{\partial \theta}{\partial t} + u_j \frac{\partial \theta}{\partial x_j} = \kappa \frac{\partial^2 \theta}{\partial x_j \partial x_j} . \tag{1}$$

The velocity field $u_j(\boldsymbol{x}, t)$ is governed by the Navier-Stokes equation,

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} \quad (i = 1, 2, 3), \tag{2}$$

and the equation of continuity,

$$\frac{\partial u_j}{\partial x_j} = 0 , \tag{3}$$

where t is the time, $\boldsymbol{x}$ is the space coordinate, p is the pressure, ρ is the constant density, ν is the kinematic viscosity of fluid, and κ is the diffusion coefficient of the passive scalar. The summation convention is assumed for repeated subscripts.

The statistical properties of this system are different depending on the Schmidt number $s = \nu/\kappa$. It is phenomenologically [1-4] predicted that the passive scalar spectral function $\Theta(k, t)$ defined by

$$\Theta(k, t) = k^2 \oint \mathrm{d}\Omega \tilde{Z}(\boldsymbol{k}, t), \tag{4}$$

where $\tilde{Z}(\boldsymbol{k}, t)$ is the Fourier component of the one-time correlation function of the scalar field $\theta(\boldsymbol{x}, t)$ and $\oint \mathrm{d}\Omega$ denotes a solid angle integration in the wavenumber space, obeys different scaling laws depending on the Schmidt number in the large wavenumber range: The spectrum consists of a scaling function proportional to $k^{-5/3}$ followed by an exponentially decay for moderate s, and the k^{-1} and the $k^{-17/3}$ scaling ranges appear between the $k^{-5/3}$ power law and the exponentially decaying ranges in the large and the small Schmidt number limits, respectively (see (13)–(15) below).

The main purpose of this paper is to show analytically such a Schmidt number dependence of the statistics of passive scalar fields based upon the

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basic equations (1)–(3) by developing the direct-interaction approximation (DIA) [5,6].

2. Lagrangian DIA

2.1 Sparse nonlinear couplings

For simplicity of the analysis, we consider first the motion of a fluid confined in a periodic cube of side L, which will be set at infinity at the final stage of formulation. Then, we rewrite the Navier-Stokes equation (2) in terms of the Fourier coefficients $\tilde{u}_i(\boldsymbol{k}, t)$ of the velocity field as

$$\left[\frac{\partial}{\partial t} + \nu k^2\right]\tilde{u}_i(\boldsymbol{k}, t) = M_{ijm}(\boldsymbol{k}) \sum_{\boldsymbol{p}} \sum_{\boldsymbol{q}} {}_{(\boldsymbol{k}+\boldsymbol{p}+\boldsymbol{q}=\boldsymbol{o})} \tilde{u}_j(-\boldsymbol{p}, t)\tilde{u}_m(-\boldsymbol{q}, t) \tag{5}$$

with

$$M_{ijm}(\boldsymbol{k}) = -\frac{\mathrm{i}}{2}\left(\frac{2\pi}{L}\right)^3 \left[k_m \tilde{P}_{ij}(\boldsymbol{k}) + k_j \tilde{P}_{im}(\boldsymbol{k})\right], \tag{6}$$

where $\tilde{P}_{ij}(\boldsymbol{k}) = \delta_{ij} - k_i k_j/k^2$. The right-hand side of (5) consists of a large number of terms, each of which represents a nonlinear interaction between three Fourier modes $\tilde{u}_i(\boldsymbol{k})$, $\tilde{u}_i(\boldsymbol{p})$ and $\tilde{u}_i(\boldsymbol{q})$. We call this nonlinear interaction appearing explicitly in the governing equation the direct interaction between these three modes. Direct interactions between $\tilde{u}_i(\boldsymbol{k}_1)$ and the other modes are depicted schematically in Fig. 1. There is only a single direct interaction between an arbitrary pair of the Fourier modes. This property, the sparseness of the nonlinear couplings, arises from the constraint $\boldsymbol{k} + \boldsymbol{p} + \boldsymbol{q} = \boldsymbol{o}$ in the summation with respect to $\boldsymbol{p}$ and $\boldsymbol{q}$ on the right-hand side of (5), and is essential in the formulation of DIA.

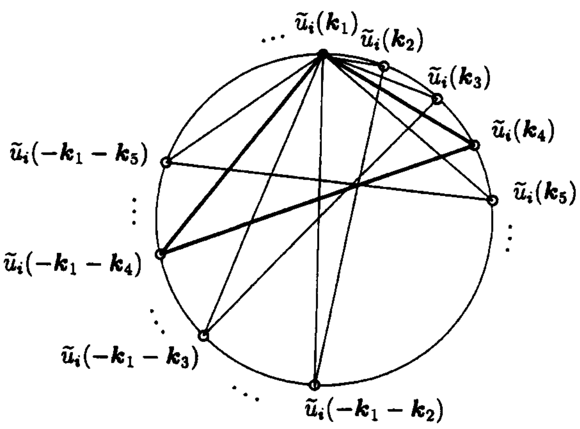

Fig. 1 Direct interactions between $\tilde{u}_i(\boldsymbol{k}_1)$ and other Fourier modes in the Navier-Stokes system, which are expressed by triangles made by three Fourier modes. We can see that there is only a single direct interaction between $\tilde{u}_i(\boldsymbol{k}_1)$ and any other modes. In this sense, this is a dynamical system with sparse nonlinear couplings. The DIA assumptions are explained as follows: Although a removal of the triad interaction shown with thick lines, for example, makes the three Fourier modes $\tilde{u}_i(\boldsymbol{k}_1)$, $\tilde{u}_i(\boldsymbol{k}_4)$ and $\tilde{u}_i(-\boldsymbol{k}_1 - \boldsymbol{k}_4)$ have no correlation, the magnitude of the difference between the true field and the fictitious field without this direct interaction remains much smaller than that of the true field.

2.2 Assumptions

The DIA is formulated under the following two assumptions. [i] The correlation between a triplet of Fourier modes is assumed to originate mainly from their direct interaction. If we removed the direct interaction between three modes, then they would become statistically independent of each other. [ii] Such an artificial removal of only one direct interaction from the system might cause very small effects to the entire statistics because there are a very large number of interactions in a high Reynolds-number system. The second assumption has only to hold during the time scale of the auto-correlation function of the velocity field. The above two assumptions may be justified for systems with a large number of degrees of freedom [7].

2.3 Langrangian fields

There have been proposed several DIA theories depending on the choice of the statistical quantities for which a closed set of equations is constructed, e.g., Kraichnan's original DIA [5] and the Lagrangian history DIAs [8]. The present formulation, described in detail in Ref. [6], is different from them in the sense that it is formulated with a set of two-time Lagrangian velocity correlation and the Lagrangian response functions, which are Galilean invariant. This Lagrangian statistical quantities were first proposed by Kaneda [9]. Using the Lagrangian position function $\psi(\boldsymbol{x}, t \mid \boldsymbol{x}', t')$ governed by

$$\frac{\partial \psi}{\partial t} + u_j \frac{\partial \psi}{\partial x_j} = 0 \tag{7}$$

with the initial condition $\psi(\boldsymbol{x}, t \mid \boldsymbol{x}', t') = \delta^3(\boldsymbol{x} - \boldsymbol{x}')$, we can relate Lagrangian fields to Eulerian counterparts. For instance, the Lagrangian velocity $v_i(t \mid \boldsymbol{x}, t')$, the velocity of a fluid particle at time t which passed at position $\boldsymbol{x}$ at time $t'(\le t)$, is expressed as

$$v_i(t \mid \boldsymbol{x}', t') = \int \mathrm{d}^3\boldsymbol{x}\, u_i(\boldsymbol{x}, t)\, \psi(\boldsymbol{x}, t \mid \boldsymbol{x}', t'). \tag{8}$$

We apply the DIA assumptions to the Eulerian fields, which have the sparse nonlinear couplings property, and derive a closed set of equations for the Lagrangian field correlation and response functions by the help of relations between the two kinds of fields such as (8).

2.4 Application to a passive scalar field

In order to apply the Lagrangian DIA to a passive scalar field, we introduce, similarly to (8), the Lagrangian passive scalar field,

$$\theta^{(L)}(t|\boldsymbol{x}', t') = \int \mathrm{d}^3\boldsymbol{x}\, \theta(\boldsymbol{x}, t)\, \psi(\boldsymbol{x}, t|\boldsymbol{x}', t'). \qquad (9)$$

A closed set of equations for the correlation and the response functions of the Fourier component of $\theta^{(L)}$ are then derived under the two assumptions described in §2.2.

By a straightforward calculation [10] from the basic equation (1) we can derive an evolution equation for the passive scalar correlation function, while the response function can be analytically solved. When the velocity and the scalar fields are statistically homogeneous, isotropic and stationary, the resultant closure equation is written as

$$\hat{\Theta}(\hat{k}) = \frac{K}{2} s \hat{k}^{-4/3} \iint_{\Delta 1} \mathrm{d}p\mathrm{d}q\, \sigma(1, p, q) p^{-8/3} q \times \left[-\hat{\Theta}(\hat{k}) + \frac{1}{q^2} \hat{\Theta}(\hat{k}q) \right] \times \int_0^{\infty} \mathrm{d}t\, Q(\hat{k}p, tp^{2/3}) \exp\left[-s^{-1} \hat{k}^{4/3} (1+q^2) t \right], \qquad (10)$$

where $\hat{k} = k/(\varepsilon^{1/4} \nu^{-3/4})$ is the nondimensional wavenumber, $\hat{\Theta} = \Theta/(\chi \varepsilon^{-3/4} \nu^{5/4})$ is the nondimensional spectral function,

$$\chi = 2\kappa \int_0^{\infty} \mathrm{d}k k^2 \Theta(k) \qquad (11)$$

is the scalar dissipation rate, ε is the energy dissipation rate, and

$$\sigma(k, p, q) = \frac{(k+p+q)(k+p-q)(k-p+q)(-k+p+q)}{4p^2 k^2}. \qquad (12)$$

The function Q is the Lagrangian velocity correlation function, which has already been known in the framework of the Lagrangian DIA for isotropic turbulence [6]. Thus, it is only the Schmidt number that characterizes the resultant closure equation (10) for the passive scalar spectrum.

3. Solution to the Closure Equation

It is shown [10-12] that the closure equation (10) has solutions which are completely consistent with phenomenologies. It is

$$\Theta(k) = C_1 \chi \varepsilon^{-1/3} k^{-5/3} \qquad (13)$$

in the inertial-advective range ($k \ll \min\{k_K, k_C\}$, Obukhov [1] and Corrsin [2]),

$$\Theta(k) = C_2 \chi \nu^{1/2} \varepsilon^{-1/2} k^{-1} \qquad (14)$$

in the viscous-advective range ($k_K \ll k \ll k_B$ for $s \gg 1$, Batchelor [3]) and

$$\Theta(k) = C_3 \chi \kappa^{-3} \varepsilon^{2/3} k^{-17/3} \qquad (15)$$

in the inertial-diffusive range ($k_C \ll k \ll k_K$ for $s \ll 1$, Batchelor, Howells & Townsend [4]. Here, the characteristic wavenumbers k_K (Kolmogorov wavenumber), k_C (Obukhov-Corrsin wavenumber) and k_B (Batchelor wavenumber) are respectively defined by

$$k_K = (\varepsilon / \nu^3)^{1/4}, \qquad (16)$$

$$k_C = (\varepsilon / \kappa^3)^{1/4} = s^{3/4} k_K \qquad (17)$$

and

$$k_B = (\varepsilon / \nu\kappa^2)^{1/4} = s^{1/2} k_K. \qquad (18)$$

4. Concluding Remarks

The Lagrangian DIA we have described here is a quite simple closure theory applied to strong turbulence, which is formulated under clear working assumptions. The resultant closed system of equations has solutions which are completely consistent with well-known phenomenologies of small-scale statistics of turbulence by Kolmogorov [13], Obukhov [1], Corrsin [2], Batchelor [3] and Batchelor et al. [4].

There are many attempts to explain DIA by Reynolds number expansions (called renormalized expansions) [14,15] or a diagram technique [16]. Even if they yield a same set of closure equation, they do not seem to capture the essence of DIA [5]. The DIA is a theory for strong nonlinear systems, in which each one of the strong nonlinear terms is treated as a perturbation instead of the whole of the nonlinear terms as done in other theories.

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