J. Plasma Fusion Res. SERIES, Vol. 2 (1999) 335–338

Long Term Evolution of Drift Vortex Structures

GOLOBOROD'KO Viktor Ya. and TARANOV Volodymyr B.
Institute for Nuclear Research, prospekt Nauki 47, Kiev-22, Ukraine

(Received: 7 December 1998 / Accepted: 5 August 1999)

Abstract

We obtained perturbation theory solution for a vortex lattice in the framework of the Hasegawa-Mima model for small but finite amplitudes of vorticity. For bigger amplitudes we elaborated relatively simple but fast and exact numerical method for the simulation of long term evolution of drift vortex structures. We carried out numerical simulations of the evolution of symmetric and antisymmetric vortex lattices for many linear wave periods. Periodic and quasi periodic regimes were observed for small initial amplitudes. Quasi chaotic generation of higher harmonics was observed for bigger amplitudes. It was also emphasized that the neglect of the linear dispersion term in stability problems of vortex structures is incorrect even for large vortex amplitudes.

Keywords:
plasma, waves, vortices, numerical simulations, nonlinear problems

1. Introduction

Waves of the vorticity, monopole and dipole vortices, vortex chains and lattices play an important role in transport processes and turbulence in a magnetized plasma as well as in planet atmospheres. Their evolution is governed by different modifications of the Hasegawa-Mima (HM) model, which features and applications were reviewed recently in [1]. In the Section 2 the HM model for drift vortex structures in plasmas and its symmetry properties important for numerical simulations are shortly reviewed. It is also discussed the possibility of neglecting linear or nonlinear terms in this model. In the Section 3 perturbation theory solution for the antisymmetric lattice of vortices of small but finite amplitude is presented. In the Section 4 our relatively simple but sufficiently exact and fast code for solving numerically Cauchy problems for the HM model is shortly reviewed. Conclusions are made in the Section 5.

2. Model

Let us consider a plasma which is inhomogeneous in x-direction (L being the characteristic inhomogeneity length) in an external magnetic field of intensity $\boldsymbol{B}$ directed along the z-axis. For low frequency perturbations we can treat electrons as massless fluid and use quasi-neutrality condition. In this way we obtain the following expression for dimensionless electron and ion fluid densities, $n = \exp(\Phi + \varepsilon x)$, where Φ is the dimensionless electrostatic potential $e\Phi/T_e$, T_e is the electron temperature, ε is a small parameter which allows to obtain HM model equations. Ion cyclotron frequency $\omega_B = eB/Mc$ and ion sound speed $c_s = (T_e/M)^{1/2}$ determine characteristic dispersion length $r_B = c_s/\omega_B$, the small parameter is equal to the ratio r_B/L. Ion fluid equations in the low frequency limit lead to the equations for the electrostatic potential Φ and ion fluid velocity $\boldsymbol{v}$. In the first order in ε we obtain

$$\nabla\cdot \boldsymbol{v}_1 = 0,\ \boldsymbol{v}_1 \times \boldsymbol{e}_z = \nabla\Phi_1,$$
$$w_1 = (\boldsymbol{e}_z \times \nabla)\Phi_1,\ \nabla \times \boldsymbol{v}_1 = \boldsymbol{e}_z \Delta\Phi_1.$$

In the second order in ε we obtain from the compatibility condition the HM model equations (index 1 of perturbation theory is omitted below):

Corresponding author's e-mail: vtaranov@kinr.kiev.ua

$$\partial\Psi/\partial t + J(\Phi, \Psi) = \partial\Phi/\partial y,\ \Psi = \Phi - \Delta\Phi$$

where Φ is the electrostatic potential and Ψ is potential vorticity. $J(\Phi, \Psi) = \partial\Phi/\partial x \cdot \partial\Psi/\partial y - \partial\Psi/\partial x \cdot \partial\Phi/\partial y$. The HM equations are written now in dimensionless variables $\varepsilon\omega_B t$, x/r_B, y/r_B, $e\Phi/T_e\varepsilon$.

The Jacobian nonlinear term in the HM model is a very particular one. For small amplitudes we can neglect this term and restrict ourselves to the analytic solutions of the linear theory. But the neglect of the linear term $\partial\Phi/\partial y$ for large amplitudes of Φ, Ψ in the stability problems is incorrect. The value of the Jacobian $J(\Phi, \Psi)$ depends not only on the amplitudes of Φ and Ψ, but also on their possible mutual functional dependence. Neglecting the linear term in advance, as it is done, e.g. in numerical simulations of [2], we approach to the stable state with tending to zero nonlinear term, so at the final stage linear dispersion term becomes important and cannot be neglected. Such numerical simulations don't reproduce the physical effect of the dispersion of vortex structures due to the emission of drift waves and this stability analysis is physically incorrect.

The HM model is invariant to the inversion of the time and of the drift direction $t \to -t$, $y \to -y$ and to the simultaneous inversion of the inhomogeneity gradient, the electrostatic potential and the vorticity $x \to -x$, $\Phi \to -\Phi$, $\Psi \to -\Psi$. Due to the last symmetry initially antisymmetric in the x direction solution remains antisymmetric, but the symmetric one looses its symmetry. In the limit of very small amplitudes symmetric solutions preserve their symmetry since linearized equations are invariant to both $x \to -x$ and $\Phi \to -\Phi$, $\Psi \to -\Psi$ transformations. This holds also for finite amplitude solutions with zero Jacobian nonlinear term, e.g. for the evolution of initially point-sized vortex.

3. Perturbation Theory

For small but finite amplitudes we obtained solutions to the HM equations by multiple-time scale formalism. We considered double periodic lattice of vortices. So our initial conditions had the form (α is the characteristic amplitude):

$$\Psi = \alpha \sin x\,(1 + \sin y).$$

For small but finite amplitude we obtained solutions of the form

$$\Psi = \alpha(\Psi_0 + \alpha\Psi_1 + ...),$$

$$\partial/\partial t = \partial/\partial t_0 + \alpha\partial/\partial t + ...,\ t_n = \alpha^n t.$$

In this way we obtained in zero approximation

$$\Psi_0 = \sin x\,(1 + \sin\,(y + t_0/3)).$$

In the first approximation (resonant terms are absent in this case) we obtained:

$$\Psi_1 = 0.5 \sin 2x\,(\sin\,(y + t_0/6) - \sin\,(y + t_0/3)).$$

The second order solution is as follows:

$$\begin{aligned}\Psi_2 = {} & 1/8\,(1 - \cos\,(t_0/6)) \sin x - \\ & 3/8\,(1 - \cos\,(t_0/6)) \sin 3x + \\ & 1/2 \sin x\,(\sin\,(y + t_0/6) - \sin\,(y + t_0/3)) + \\ & 3/8 \sin x\,(\cos\,(2y + t_0/2) - \cos\,(2y + t_0/3)) + \\ & 3/16 \sin x\,(\cos\,(2y + t_0/3) - \cos\,(2y + 2t_0/3)) + \\ & 11/10 \sin 3\,(\sin\,(y + t_0/11) - \sin\,(y + t_0/6)) + \\ & 11/32 \sin 3x\,(\sin\,(y + t_0/3) - \sin\,(y + t_0/11)) + \\ & 7/120 \sin 3x\,(\cos\,(2y + t_0/7) - \cos\,(2y + t_0/2)) + \\ & 7/176 \sin 3x\,(\cos\,(2y + 2t_0/3) - \cos\,(2y + t_0/7)).\end{aligned}$$

Cancelling the resonant term in the second order solution, we determine the nonlinear frequency shift $\alpha^2/12$ in the Ψ_0 term:

$$\Psi_0 = \sin x\,(1 + \sin\,(y + t_0/3 + t_2/12)).$$

4. Numerical Scheme

For larger amplitudes $\alpha \geq$ the perturbation theory doesn't work and we carried out numerical simulations. On each time step we evaluated the electrostatic potential Φ from known values of the vorticity Ψ. Itís useful to do this by the Fast Hartley Transform (FHT) instead of the Fast Fourier Transform (FFT). The FHT needn't the use of complex variables, coincides with its inverse transform and is as fast as the FFT. The FHT transforms of Φ and Ψ are as follows:

$$\Phi_{kl} = 1/N\, \Sigma\, A_{k'l'} \,\mathrm{cas}\,(2\pi kk'/N)\, \mathrm{cas}\,(2\pi ll'/N),$$
$$\Psi_{kl} = 1/N\, \Sigma\, B_{k'l'} \,\mathrm{cas}\,(2\pi kk'/N)\, \mathrm{cas}\,(2\pi ll'/N),$$

where $\mathrm{cas}\ \theta \equiv \cos\theta + \sin\theta$, the summation over k', l' is performed from 0 to $N - 1$. For this 2D FHT we used the direct product of two 1D FHT presented in [3] which we modified to the faster form by storing index permutations in the computer memory in advance and combining the FHT transform with the calculation of the linear part of the evolution operator.

So we obtain Φ from Ψ by the division $A_{kl} = B_{kl}/D_{kl}$ with the scalar operator D defined as

$$D_{kl} = 1 + (N/\pi)^2 (\sin^2 (\pi k/N) + \sin^2 (\pi l/N)),$$

which is the same as the corresponding FFT operator. But to reproduce the linear evolution part $\partial\Psi/\partial t = \partial\Phi/\partial y$ of the HM model we used more complicated operator $S_{kl} = \sin(2\pi l/N)/(2\pi/N)D_{kl}$, so the linear evolution is reproduced by the formulae

$$B_{k0} = B_{k0}(0),$$
$$B_{kl} = B_{kl}(0) \cos (S_{kl}\tau) - B_{k,N-1}(0) \sin (S_{kl}\tau), \dots$$
$$B_{k,N-1} = B_{k,N-1}(0) \cos (S_{k,N-1}\tau) - B_{kl}(0) \sin (S_{k,N-1}\tau)$$

where τ stands for the time step, in our case $\tau = 2\pi/N$, $N = 256$.

Different initial conditions were considered, namely the antisymmetric in the inhomogeneity profile direction Ox

$$\Psi(t = 0, x, y) = \alpha \sin x (1 + \sin y),$$

the symmetric conditions with nonzero mean vorticity, ($m = 1, \dots, 8$):

$$\Psi(t = 0, x, y) = \alpha ((1 - \cos x)/2)^m((1 - \cos y)/2)^m$$

and the point-sized initial disturbations of the vorticity.

For the modelling of the Jacobian nonlinear term we used the Arakawa scheme [4]. We omitted in this scheme terms which mutually cancel and obtained the procedure of evaluation of the Jacobian W of two arrays F and G on the $N \times N$ grid illustrated below by the fragment of the corresponding subroutine

```
ZET = (F(I, JM) + F(IP, JM) - F(I, JP)
      - F(IP, JP))*G(IP, J)
TEZ = (F(IM, JM) + F(I, JM) - F(IM, JP)
      - F(I, JP))*G(IM, J)
ZTE = (F(IP, J) + F(IP, JP) - F(IM, J)
      - F(IM, JP))*G(I, JP)
TZE = (F(IP, JM) + F(IP, J) - F(IM, JM)
      - F(IM, J))*G(I, JM)
ZUB = ZET - TEZ + ZTE - TZE
ZET = (F(IP, J) - F(I, JP))*G(IP, JP)
TEZ = (F(I, JM) - F(IM, J))*G(IM, JM)
ZTE = (F(I, JP) - F(IM, J))*G(IM, JP)
TZE = (F(IP, J) - F(I, JM))*G(IP, JM)
ZUB = ZUB + ZET - TEZ + ZTE - TZE
W(I, J) = ZUB
```

where IP stands for $I + 1$, IM for $I - 1$ and so on. Obtained in this way values must be divided by $12\ h^2$, h is the grid step which in our calculation was equal to the time step $2\pi/256$. Time evolution was realized by the leapfrog scheme. Program was written in FORTRAN-77 and run on the Pentium 166 MMX. Machine time for one time step was around 2 seconds on the 256×256 grid. Energy, $\Phi\Psi dxdy$, and mean square vorticity $\int\Psi^2 dxdy$ integrals were conserved with less than one percent error for thousands of time steps. So our relatively simple code seems to be sufficiently fast and exact one.

For the point vortex initial condition numerical simulation reproduce anisotropic and non-self-similar Green function of the linear theory, since the nonlinear

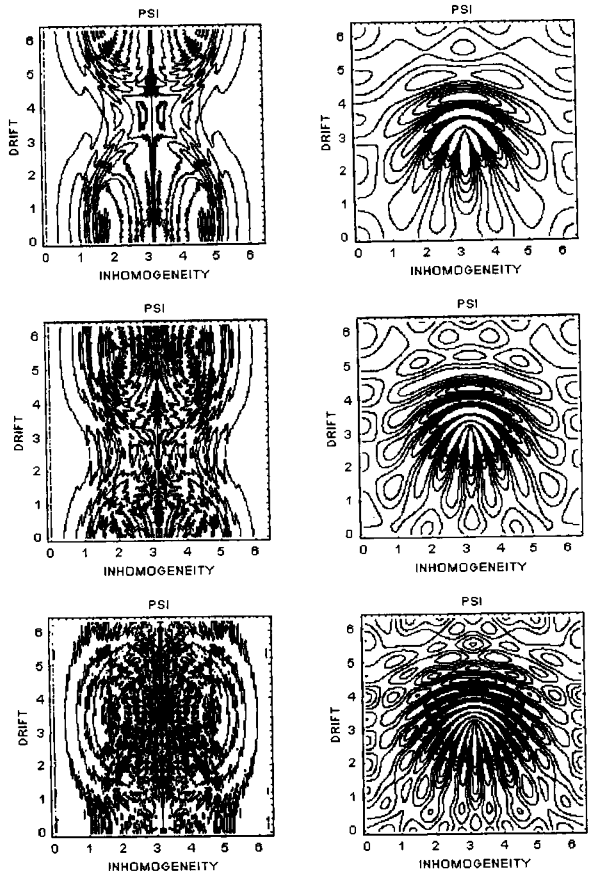

Fig. 1 Contour plots of the vorticity describing temporal evolution of an antisymmetric vortex lattice with amplitude $\alpha = 1$ (left column) and of an initially point-sized vortex (right column). Time values are 12π, 24π and 48π (from the top to the bottom).

term exactly vanish in this case. Contour plots of the vorticity are presented in the right column of Fig. 1. Values of time are equal to 12π, 24π and 48π (from the top to the bottom). For symmetric and antisymmetric lattices we obtained periodic and quasiperiodic solutions for $\alpha \le 0.5$. Antisymmetric solutions were in a good agreement with our perturbation theory for $\alpha \le 0.25$. Quasi chaotic generation of many higher harmonics in the time of few linear wave periods was observed for $\alpha \ge 0.5$. Contour plots of the vorticity for the antisymmetric lattice with $\alpha = 1$ are presented in the left column of Fig. 1.

5. Conclusions

Linear dispersion term cannot be neglected in the HM model for drift vortex structures in plasmas. So the results of the numerical simulations [2] are applicable only to the homogeneous plasma or to the top domain of density profiles like $n = \exp(\Phi - \varepsilon^2 x^2)$, since for such profiles inhomogeneity effects are present only in the order ε^2, but the HM model is the first order in e theory.

Combining rationalized Arakawa representation for the Jacobian nonlinear term with Fast Hartley Transform for calculations of the potential Φ from the known vorticity Ψ and of the linear evolution part of the HM model, we obtained relatively simple but sufficiently stable, fast and exact numerical code. This code allowed us to perform many numerical simulations of the temporal evolution of vortex lattices and initially point-sized vortices for large time intervals of many linear wave periods. We observed vortex latticesí in small amplitude periodic and quasiperiodic regimes, described also by our multiple-time-scale perturbation theory solution, and quasichaotic generation of many harmonics when the dimensionless amplitude was of order 1.

References

[1] W. Horton and A. Hasegawa, Chaos **4**, 227 (1994).

[2] N. Kukharkin *et al.*, Phys. Rev. Lett. **75**, 2486 (1995).

[3] R. Bracewell, *The Hartley Transform* (Oxford Univ. Press, N.Y., 1986).

[4] A. Arakawa, J. Comput. Phys. **1**, 119 (1966).