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Theoretical Study of Inhomogeneous Plasma Turbulence

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Abstract

A hierarchy of models, *i.e.*, (1) a model with many degrees of freedom, (2) a model with intermediate degrees of freedom, (3) a model with a few degrees of freedom is considered to understand the nature of turbulence. Results obtained from models of different levels are compared and characteristics of the interchange mode turbulence such as chaotic nature, cascade and statistical expression are discussed.

Keywords:
plasma turbulence, anomalous transport, nonlinear simulation, Lyapunov exponent

1. Introduction

Much work has been done in the research of inhomogeneous plasma turbulence and associated transport. Recent research shows, however, nonlinear instabilities play an important role in the transport driven by the inhomogeneous plasma turbulence rather than linear instabilities [1]. The analysis of subcritical turbulence brings us a new insight for plasma turbulence. As a typical example of the inhomogeneous plasma turbulence, the interchange mode turbulence in a system with the inhomogeneities of pressure and magnetic field is considered. Based on the direct simulation of the reduced MHD equations [2], it is shown that (1) the nonlinear plasma turbulence develops even in a linearly-stable region, (2) the threshold value of the fluctuation amplitude for the nonlinear instability is much lower than the stationary level (3) the strong turbulent state is self-sustained and the fluctuation amplitude is proportional to the power of the pressure gradient. These results indicate that the strong turbulence is sustained by the inhomogeneities and the conventional method which is based on elaborate evaluation of the linear growth rate is insufficient.

The nonlinear destabilization mechanism discussed here originates from the anomalous electron viscosity due to the energy cascades to short wavelength modes. This mechanism also leads to the instability of the trajectory in phase space. Positive Lyapunov exponents and short correlation time are observed in this system. It is a challenging problem to understand the relation between the correlation time and the irreversibility due to global transport. In the study of turbulence, a system with high Reynolds number is important. However, it is difficult to solve the equations in the inviscid limit even by a super-computer. In order to understand the issues above we consider a hierarchy of the models, *i.e.*, (1) a model with many degrees of freedom like direct simulation [3], (2) a model with intermediate degrees of freedom such as a shell model [4], (3) a model with a few degrees of freedom such as the Lorenz model [5]. Results obtained from models of different levels are compared and characteristics of the interchange mode turbulence such as chaotic nature, cascade and intermittency are discussed.

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2. Model with Many Degrees of Freedom

We here adopt the three field reduced set of equations in a shear magnetic field with bad averaged curvature. This model is relevant to study the nonlinear dynamics and associated transport by the electrostatic current diffusive interchange mode turbulence in a system of magnetic hill. The equation of motion, Ohm's law and the energy balance equation are given in a normalized form as

$$\frac{\mathrm{d}}{\mathrm{d}t}\nabla_\perp^2\phi = ik_y\, sxj - j\alpha k_y\, p + \mu_c \nabla_\perp^4 \phi, \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} j = -ik_y\, sx\phi + \lambda_c \nabla_\perp^2 j, \tag{2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} p = -ik_y\, \phi + \chi_c \nabla_\perp^2 p, \tag{3}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + [\phi,\ ]$, [,] is the Poisson bracket denoting the $\boldsymbol{E}\times\boldsymbol{B}$ nonlinearity. For the normalization, the poloidal Alfven time and the electron skin depth are employed (see Ref. [2] for details). Figure 1 shows the temporal evolution of the growth of fluctuating pressure energy (nearly equal to total energy), which demonstrates the nonlinear growth of fluctuations. We choose the parameters $\mu_c = \chi_c = 0.2$, $\lambda_c = 0.01$ and $s = \alpha = 0.5$. The solid line is the result of the solution of Eqs. (1)-(3) and the electron nonlinearity in Eq. (2) is kept. (Dashed line: the case without the nonlinearity in Eq. (2).) In the small amplitude limit, the perturbations grow following the linear growth rate. However, at the time of $t \simeq 10$, when the amplitude exceeds a certain threshold value, the growth rate starts to increase. In the time range of $10 \le t \le 60$, the growth rate becomes larger as the amplitude increases. This shows the nonlinear destabilization originated from the electron dynamics. During this phase, the normal cascade to higher k_y mode develops. It is found that the saturation amplitude is 3 times higher than that of the case without electron nonlinearity in Eq. (2). An instantaneous maximum Lyapunov exponent is evaluated from the relation $\delta E \propto e^{2\lambda t}$, where λ represents the instantaneous maximum Lyapunov exponent and δE is given by

$$\delta E = \frac{1}{L}\int_{-L_x/2}^{L_x/2} \mathrm{d}x\, \frac{1}{2}\sum_{k_y}\left(\left|\nabla_\perp \delta\phi\right|^2 + \left|\delta j\right|^2 + \left|\delta p\right|^2\right)$$

with

$$\frac{\partial}{\partial t}\nabla_\perp^2 \delta\phi + [\delta\phi, \nabla_\perp^2\phi] + [\phi, \nabla_\perp^2 \delta\phi]$$
$$= ik_y\, sx\delta j - j\alpha k_y\, \delta p + \mu_c \nabla_\perp^4 \delta\phi,$$
$$\frac{\partial}{\partial t}\delta j + [\delta\phi, j] + [\phi, \delta j] = ik_y\, sx\delta\phi + \lambda_c \nabla_\perp^2 \delta j,$$

and

$$\frac{\partial}{\partial t}\delta p + [\delta\phi, p] + [\phi, \delta p] = ik_y\, sx\delta\phi + \chi_c \nabla_\perp^2 \delta p.$$

Figure 2 shows the time evolution of l. For comparison, total energy defined by

$$E_T = \frac{1}{L}\int_{-L_x/2}^{L_x/2} \mathrm{d}x\, \frac{1}{2}\sum_{k_y}\left(\left|\nabla_\perp \phi\right|^2 + \left|j\right|^2 + \left|p\right|^2\right),$$

is also plotted. In the calculation, we renormalize the amplitudes $\delta\phi$, δj, δp at each time step of $\Delta t = 30$. It is observed that at $t < 200$, the behavior of the instantaneous maximum Lyapunov exponent is similar to the fluctuation amplitude ϕ, j, p. However, at $t > 200$, the behavior is different from that of the total energy. The value of the maximum Lyapunov exponent is obtained as order unity. This system describes the submarginal turbulence sustained by the nonlinear instability which is associated with the strong normal cascade and the positive maximum exponent.

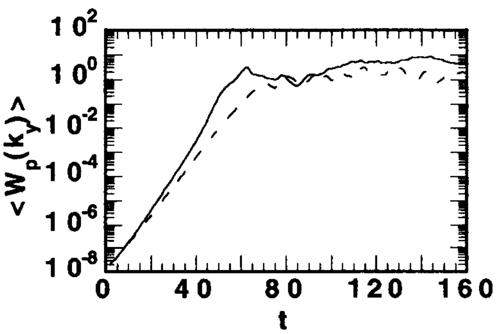

Fig. 1 Fluctuating pressure energy versus time.

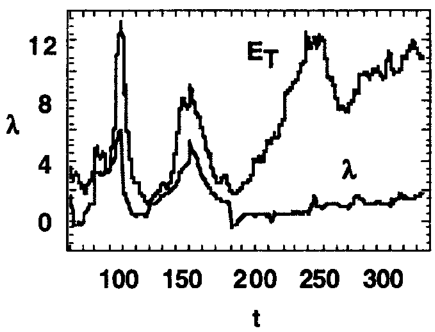

Fig. 2 Time evolution of the instantaneous maximum Lyapunov exponent.

3. Model with Intermediate Degrees with Freedom

We extend a shell model by including the thermal instability terms (Rayleigh Bénard convection) [6]. Normalized shell model is expressed in Fourier space as

$$\frac{du_n}{dt} = i(a_n u^*_{n+1} u^*_{n+2} + b_n u^*_{n-1} u^*_{n+1} + c_n u^*_{n-1} u^*_{n-2}) + P_r \theta_n - P_r k_n^2 u_n \tag{4}$$

$$\begin{aligned}\frac{d\theta_n}{dt} = i\{ & e_n (u^*_{n-1} \theta^*_{n+1} - u^*_{n+1} \theta^*_{n-1}) \\ & + g_n (u^*_{n-2} \theta^*_{n-1} + u^*_{n-1} \theta^*_{n-2}) \\ & + h_n (u^*_{n+1} \theta^*_{n+2} + u^*_{n+2} \theta^*_{n+1}) \\ & + R_a u_n - k_n^2 \theta_n\end{aligned} \tag{5}$$

where * represents the complex conjugate, $n = 1, \ldots, N$, number of each shell, u_n, the fluctuating velocity, θ_n, the fluctuating temperature, $k_n = C2^{n-1}$, the wave number, P_r, the Prandtl number, R_a, the Rayleigh number. For the boundary conditions, we assume $u_0 = u_{N+1} = 0$ and $\theta_0 = \theta_{N+1} = 0$ and the coefficients are given by

$$a_n = k_n, \quad b_n = -\frac{k_{n-1}}{2}, \quad c_n = -\frac{k_{n-2}}{2},$$
$$e_n = \frac{k_n}{2}, \quad g_n = -\frac{k_{n-1}}{2}, \quad h_n = -\frac{k_{n+1}}{2},$$

and $b_1 = b_N = c_1 = c_2 = a_{N-1} = a_N = e_1 = e_N = g_1 = g_2 = h_{N-1} = h_N = 0$. Here the typical scale length and the thermal diffusion time are used for normalization [7]. We use the following parameters: $k_1 = C = 10^{-2}$, $P_r = 1$, $R_a = 10^4$, $N = 20$ for calculations. Figure 3 shows the distribution of instantaneous Lyapunov exponents at $t = 1$. It is found that first 12 Lyapunov exponents are positive for the system with mode numbers $N = 20$, where total Lyapunov exponents are given by 80 in this system and the maximum Lyapunov exponent is obtained as 2290. Figure 4 shows the scaling exponent ζ_p for the structure function versus order p. The structure functions are calculated by $S^u_p(n) = \langle |u_n|^p \rangle$ and $S^\theta_p(n) = \langle |\theta_n|^p \rangle$, where the bracket represents the time average in the intervals with $t = 0.2 - 10$. The scaling exponents are defined by $\langle |u_n|^p \rangle \propto k_n^{-\zeta^u_p}$ and $\langle |\theta_n|^p \rangle \propto k_n^{-\zeta^\theta_p}$. The circle and the triangle correspond to ζ^u_p and ζ^θ_p, respectively. For comparison, the result obtained by GOY model [8] is also plotted (denoted by the square). The dashed line corresponds to K41 [9]: $\zeta_p = p/3$. Intermittency in shell models is the subject of much current work and is quite poorly understood at the moment. The scaling exponent for the structure function of fluctuating velocity field agrees with the result of GOY model, and is larger than that of fluctuating temperature field. They are not affected by the linear instability so much, but reflect the characteristics of nonlinearity. This system describes the strong turbulence with high Rayleigh number, which contains the sufficient numbers of positive Lyapunov exponents, intermittency nature and the normal cascade [9].

4. Model with a Few Degrees of Freedom

We extend Lorenz model by introducing higher Fourier harmonics which describe the shear flow effect. Normalized five components model [10] is written as

$$\frac{dX}{d\tau} = VW + P_r(Y - X), \quad \frac{dY}{d\tau} = -XZ + rX - Y,$$
$$\frac{dZ}{d\tau} = XY - bZ, \quad \frac{dV}{d\tau} = -XW - \gamma_V V,$$
$$\frac{dW}{d\tau} = -cXV - \gamma_W W \tag{6}$$

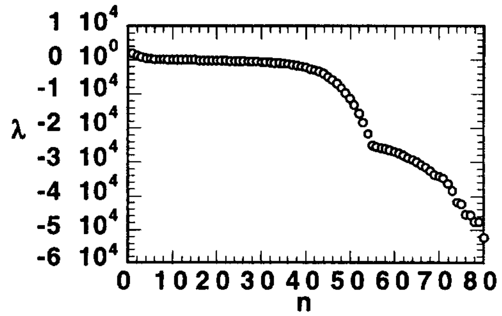

Fig. 3 Distribution of instantaneous Lyapunov exponents.

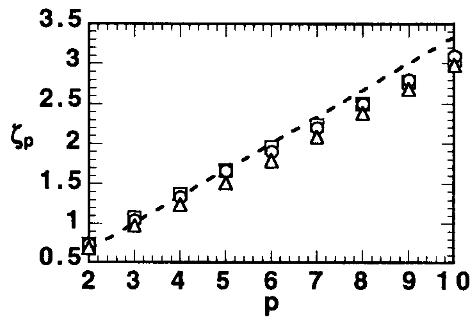

Fig. 4 Scaling exponent for the structure function versus order.

with

$$r = \frac{q^2}{\left(\pi^2 + q^2\right)^3} R_a, \quad b = \frac{4\pi^2}{\pi^2 + q^2},$$

$$\gamma_V = \frac{\pi^2}{\pi^2 + q^2} P_r, \quad \gamma_W = \frac{4\pi^2 + q^2}{\pi^2 + q^2} P_r$$

and

$$c = \frac{3q^2}{4\left(4\pi^2 + q^2\right)},$$

where q indicates (normalized) wave number of cell-arrangement. The Lorenz model is reproduced if we set $V = W = 0$. In simulations, the normalized Rayleigh number is chosen to be an order parameter to characterize the system behavior and the other parameters are fixed as $q = \pi$, $b = 20$, $\gamma_V = 10.0$, $c = 0.15$ and $P_r = 4.0$. Figure 5 shows Lyapunov exponents versus normalized Rayleigh number. It is found that for the parameter regime of $r < 140$, the similar behavior of the Lyapunov exponent is observed to that of Lorenz model. We observe the chaotic behavior and the limit cycle solution as well as the period-doubling process of thesolution in the regime of $306 < r < 2200$. For the regime of r > 2200 this model also falls into the limit cycle. This fact indicates that a few numbers of Fourier modes are not enough to describe the chaos in the regime with high Rayleigh number. However, this model is useful to study the transition to chaos from laminar state.

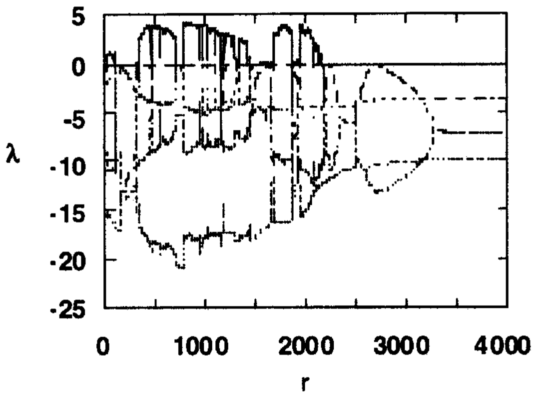

Fig. 5 Lyapunov exponents versus normalized Rayleigh number.

5. Summary and Discussions

To study the interchange mode turbulence in the various range of Rayleigh number, we introduce hierarchical model equations: (I) a model with many degrees with freedom, (II) a model with intermediate degrees of freedom, (III) a model with a few degrees of freedom. It is found that the nonlinear growth of instability is characterized by normal cascade and positive Lyapunov exponent. Intermittent nature is discussed by the higher order structure functions of a shell model. Intermittency is considered to be related with the normal cascade [9]. At this point, the relation between the submarginal instability and intermittency is not clear. It is left for a future work. The transition to chaos is discussed based on the extended Lorenz model. Various dynamical behaviors which are not described by the Lorenz model are observed in the range of $306 < r < 2200$, however, for $r > 2200$, this model is not relevant to describe chaos. The absolute value of saturation amplitude of energy or flux is predicted only by model (I), however, universal scaling law such as Kolmogorov inertia scaling law is also described by model (II). Model (II) has an advantage to investigate multi-fractal nature of turbulence, i.e., intermittency and transition from chaos to turbulence. Inter-relation between a model with a few degrees of freedom and that with intermediate or many degrees of freedom should be investigated for future.

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