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Vlasov Continuum Contribution to Anomalous Plasma Transport

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Abstract

The propagator for the Vlasov-Poisson system in inhomo-geneous plasma is applied for the spectral decomposition of the perturbed distribution function, and the anomalous cross field flux which is decomposed into three portions, the discrete mode determined by the dispersion relation, the ballistic mode, and the pure continuum contribution. The flux due to the discrete mode (instability) is ambipolar, while the fluxes due to the continuum, in general, are not ambipolar.

Keywords:

propagator, spectral decomposition, anomalous transport

1. Introduction

Particle transport process may consist of multiple scattering with medium or fields, and free streaming [1]. The former diffusion process may correspond to the discrete mode determined from the dispersion relation [2], while the free streaming may correspond to the contribution from the continuous eigenvalue of the transport equation [3]. The importance of the spectral decomposition of the transport process is that the anomalous transport induced by the discrete mode (instability) is ambipolar, while that induced by the continuum is not, i.e., the latter may be responsible to the generation of the radial electric field. In this paper, a previous attempt to apply the spectral decomposition to anomalous plasma transport flux [4] is reformulated making use of the propagator for the Vlasov-Poisson system in an inhomogeneous toroidal plasma.

2. Propagator for Vlasov-Poisson System

First we consider the Vlasov equation

$$\frac{\partial f}{\partial t} + Lf = 0 \tag{1}$$

where the Louville operator L is defined by

$$L = \boldsymbol{v}\cdot\nabla + \frac{1}{m}\boldsymbol{F}\frac{\partial}{\partial \boldsymbol{v}} \tag{2}$$

The Lorenz force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = e\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B} \tag{3}$$

The solution to Eq. (1) is formally given by

$$f(\boldsymbol{r}, \boldsymbol{v}, t) = U_t f(\boldsymbol{r}, \boldsymbol{v}, 0) \tag{4}$$

where the solution (semi-group) operator U is defined by

$$U_t = \exp\left(-\int_0^t \mathrm{d}tL\right) \tag{5}$$

This operator can be expanded as

$$U_t = \sum \frac{1}{n!}\left\{-\int_0^t \mathrm{d}t\left(\boldsymbol{v}\cdot\nabla + \frac{1}{m}\boldsymbol{F}\frac{\partial}{\partial \boldsymbol{v}}\right)\right\}^n \tag{6}$$

which means the Taylor expansion operator with respect to the spatial and velocity coordinates, $(\boldsymbol{r}, \boldsymbol{v})$ of the

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initial distribution function $\tilde{f}(\boldsymbol{r}, \boldsymbol{v}, 0)$. In the linear theory, the solution can be expressed by

$$f(\boldsymbol{r}, \boldsymbol{v}, t)=f\left(r-\int \mathrm{d}t\boldsymbol{v}, \boldsymbol{v}-\int \mathrm{d}t\, \frac{1}{m}\boldsymbol{F}, 0\right). \quad (7)$$

The operator U is a shift operator to the initial distribution. Denoting the spatial and velocity shifts respectively by Δr and Δv, expanding to the second order, ensemble averaging, and making time derivative, we have a transport equation

$$\frac{\partial \tilde{f}}{\partial t}=\left(\frac{\partial}{\partial r}D_{rr}\frac{\partial}{\partial r}+\frac{\partial}{\partial r}D_{rv}\frac{\partial}{\partial v}+\frac{\partial}{\partial v}D_{vv}\frac{\partial}{\partial v}\right)\tilde{f}(r, v, t), \quad (8)$$

where D_{rr}, D_{rv} and D_{vv} are diffusion coefficients, for example $D_{rr} = < (\Delta r)^2 >/2t$. The particle displacements in space and velocity may be due to random scatterings with waves, and/or free streaming. We here consider how these processes may decompose the transport quantities.

For the sake of simplicity, we assume the electrostatic approximation, and expand all perturbed quantities in Fourier series of the form

$$f(r, v, t)=\sum f(r, v, t)\exp(\mathrm{i}k.r). \quad (9)$$

If we make Laplace transform to Eq. (1) with respect to time t:

$$\hat{f}(\boldsymbol{k}, \boldsymbol{v}, \omega)=\int_0^\infty \mathrm{d}t \exp(\mathrm{i}\omega t)\tilde{f}(\boldsymbol{k}, \boldsymbol{v}, t),$$

the solution can formally be written as

$$\hat{f}(\boldsymbol{k}, \boldsymbol{v}, \omega)=(\mathrm{i}\omega-L)^{-1}\tilde{f}(\boldsymbol{k}, \boldsymbol{v}, 0).$$

In this case, the Poisson equation is written as

$$\hat{\boldsymbol{E}}=-\sum\frac{4\pi e_j}{k^2}\mathrm{i}\boldsymbol{k}\int\hat{f}_j\mathrm{d}^3v. \quad (10)$$

The Louville operator can be expressed in the form [5]

$$L_i(\boldsymbol{k}, \boldsymbol{v})=\mathrm{i}\boldsymbol{k}\cdot \boldsymbol{v}+\frac{e_i}{m_i}\boldsymbol{v}\times\boldsymbol{B}\frac{\partial}{\partial \boldsymbol{v}} -\frac{4\pi e_j}{m_i k^2}\mathrm{i}\boldsymbol{k}\frac{\partial f_i}{\partial \boldsymbol{v}}\sum e_j\int \mathrm{d}^3v. \quad (11)$$

The Laplace transformed propagator can be written as

$$\frac{1}{\mathrm{i}\omega-L}=\frac{1}{\mathrm{i}(\omega-\boldsymbol{k}.\boldsymbol{v})}\left(1-\frac{\alpha_j(\boldsymbol{k},\boldsymbol{v})}{\varepsilon(\boldsymbol{k},\boldsymbol{v})}\right)\sum e_j\int\frac{\mathrm{d}^3v}{\mathrm{i}(\omega-\boldsymbol{k}.\boldsymbol{v})}, \quad (12)$$

where α is defined by

$$\alpha_i(\boldsymbol{k}, \boldsymbol{v})=-\frac{4\pi e_j}{m_i k^2}\mathrm{i}\boldsymbol{k}\frac{\partial f_i}{\partial \boldsymbol{v}}, \quad (13)$$

and the dielectric function is given by

$$\varepsilon(\boldsymbol{k}, \omega)=1+\sum_j\chi_i(k, \omega) \quad (14)$$

where the susceptibility χ is defined by

$$\chi_i(k, \omega)=\int\frac{\mathrm{d}^3ve_j\alpha_j(k, v)}{\mathrm{i}(\omega-\boldsymbol{k}.\boldsymbol{v})}. \quad (15)$$

The propagator is analytic in the complex ω-plane except the real axis and the pole ω_0 given by the dispersion relation

$$\varepsilon(\boldsymbol{k}, \omega_0)=0 \quad (16)$$

i.e., it is discontinuous across the real axis, and has poles at $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ and $\omega = \omega_0$. The pole at $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ on the real axis gives the ballistic mode. The discontinuity of the propagator is due to the velocity integration of the ballistic mode. From the view point of the eigenvalue problem of the Vlasov equation, the propagator is convenient for the spectral decomposition of the perturbed quantities $\hat{f}$ and $\hat{\phi}$ in the inverse Laplace transformation. The pole $\omega = \omega_0$ corresponds to the discrete eigenvalue, while the pole $\omega = \boldsymbol{k}\cdot\boldsymbol{v}$ for all v corresponds to the continuous eigenvalue C.

Operating the propagator (12) to the initial distribution function $\tilde{f}(r, v, 0)$, we have the perturbed transformed distribution function $f(r, v, \omega)$. The inverse Laplace transformation for $\hat{f}(r, v, \omega)$ by

$$\tilde{f}(\boldsymbol{k}, v, t)=\frac{1}{2\pi}=\int_{-\infty+\sigma\mathrm{i}}^{\infty+\mathrm{i}\sigma}\mathrm{d}\omega \exp(-\mathrm{i}\omega t)\hat{f}(k, v, \omega). \quad (17)$$

yields the decomposed solution as shown in the next section.

If we take into account the higher order effect to the Vlasov equation (1), "collision term", which is described in terms of the two point correlation function g_{ij}, has to be included. The equation for the two point correlation function can also be solved by applying the propagator (12) [5]. In the inverse Laplace transformation of g_{ij}, it is also found that the two point correlation function can be decomposed into the discrete mode induced by the dispersion relation and the continuum contribution which is further decomposed into the ballistic mode and pure continuum contribution. In the time asymptotic limit, the ballistic mode contribution to g_{ij} gives the collision term of the Balescu-Lenard equation [6]. That is the ballistic continuum contribution determines the equilibrium distribution $<f>$ in the nonlinear asymptotic limit, although it may damp in the linear theory [7]. Since the collision term gives the anomalous transport coefficient in velocity space through the Fokker-Planck equation, the transport coefficient is also decomposed into the discrete mode and continuum contributions.

3. Cross Field Plasma Flux in Nonuniform System

We now develop the propagator to a non-uniform Maxwellian plasma in a toroidal system. If we assume the equilibrium distribution in the usual form $f_0(x, v) = N(x - v_y/\Omega) f_M(v)$ with $f_M(v)$ being the usual Maxwellian distribution function, and introducing into the gyrokinetic equation, we have

$$\frac{1}{i\omega - L(k, v)} = \frac{1}{\omega - \omega_v(k, v)} \left\{ \delta_{ij} - \frac{\alpha_j(k, v)}{\varepsilon(k, v)} \sum \int \frac{d^3 v e_j}{\omega - \omega_v(k, v)} \right\}, \quad (18)$$

where $\omega_v(k, v) = k_z v_z + \omega_D(v) - i\nu$, the coefficient α_j is

$$\alpha_j = \frac{k_D^2}{k^2} \left\{ \omega - \omega_v(k, v) - J_0^2(\omega - \omega_T{}^*) \right\} f_{Mi}(v), \quad (19)$$

ν is a nonlinear damping rate [7], $\omega_D = 2\varepsilon_n \omega^* (v_\perp^2/2 + v_z^2)$, $\omega^* = cTk_y/eBL_n$ and are the curvature drift and diamagnetic drift frequencies, respectively. The susceptibility becomes

$$\chi_i(k, \omega) = \frac{k_D^2}{k^2} \int d^3 v \left(1 - J_0^2 \frac{\omega - \omega_T^*}{\omega - \omega_v(k, v)} \right) f_{Mi}(v), \quad (20)$$

where k_D is the Debye wave number. The scalar potential ϕ is expressed by

$$\phi(k, \omega) = \frac{4\pi}{k^2} \frac{s(k, \omega)}{\varepsilon(k, \omega)}, \quad (21)$$

where s is defined by

$$s(k, \omega) = \sum_j e_i \int \frac{d^3 v \tilde{f}(k, v, 0)}{\omega - \omega_v(k, v)}. \quad (22)$$

The transformed solution $\hat{f}(k, v, \omega)$ is obtained by operating the propagator (18) to an initial distribution $\tilde{f}(k, v, 0)$. Since the propagator (18) has the poles determined from the dispersion relation (16) and $\omega = \omega_v$ embedded in the continuum C which is the set on the real axis $\omega = \omega_v(k, v)$ for all v, it is convenient for the spectral decomposition of the solution. By the inverse Laplace transformation (17), the time dependent solution can be expressed in the form:

$$\tilde{f}(k, v, t) = f_0(k, v) \exp(-i\omega_0(k)t) + f_b(k, v) \exp(-i\omega_v(k, v)t) + f_c(k, v, t), \quad (23)$$

where the first term is the discrete mode (pole contribution) determined from the dispersion relation, the second term is the ballistic mode given by the pole $\omega = \omega_v(k, v)$ and the third term is pure continuum contribution given respectively:

$$f_0(k, v) = \frac{k_D^2}{k^2} \left(1 - J_0^2 \frac{\omega_0 - \omega_T^*}{\omega_0 - \omega_v(k, v)} \right) \frac{f_M}{e} \frac{s(k, \omega_0)}{\varepsilon'(k, \omega_0)}, \quad (24)$$

$$f_b(k, v) = -\frac{1}{2} \left\{ \tilde{f}(k, v, 0) - \frac{k_D^2}{k^2} J_0^2(\omega_v(k, v) - \omega_T^*) \frac{f_M}{e} \frac{s(k, \omega_v)}{\varepsilon(k, \omega_v)} \right\}, \quad (25)$$

$$f_c(k, v, t) = -\frac{i}{2\pi} \int_C d\omega e^{-i\omega t} \frac{k_D^2}{k^2} \left(1 - J_0^2 \frac{\omega_0 - \omega_T^*}{\omega_0 - \omega_v(k, v)} \right) \frac{f_M}{e} \frac{s(k, \omega)}{\varepsilon(k, \omega)}, \quad (26)$$

where the species index has been omitted. The factor 1/2 in Eq. (25) is due to the half circle around the pole $\omega_v(k, v)$, because $\varepsilon(\boldsymbol{k}, \omega)$ and $\sigma(\boldsymbol{k}, \omega)$ are discontinuous across C, and we can not close the circle in the residue evaluation.

By the inverse Laplace transformation of Eq. (21), we have the decomposed scalar potential:

$$\phi(k, t) = \phi_0(k) \exp(-i\omega_0 t) + \phi_b(k, t) + \phi_c(k, t), \quad (27)$$

where

$$\phi_0(k) = -\frac{4\pi}{k^2} \frac{s(k, \omega_0)}{\varepsilon'(k, \omega_0)}, \quad (28)$$

$$\phi_b(k, t) = -\frac{2}{k^2} \sum_j e_j \int d^3 v \frac{\tilde{f}(k, v, 0)}{\varepsilon(k, \omega_v(k, v))} \exp(-i\omega_v(k, v)t), \quad (29)$$

$$\phi_c(k, t) = -\frac{2i}{k^2} \int_C d\omega \exp(-i\omega t) \frac{s_p(k, \omega)}{\varepsilon(k, \omega)}. \quad (30)$$

As compared with the previous study [4], the ballistic mode are separated from the continuum contributions for f and ϕ, because the ballistic contribution plays an essential role in the Balescu-Lenard collision term [6]. The ballistic mode (29) has been derived by changing the order of integrations with respect to ω and v in the function s.

Let us now evaluate the cross field plasma flux:

$$\Gamma = \int \mathrm{d}^3 v \left\langle \tilde{f}(r, v, t)\, \tilde{v}(r, t) \right\rangle . \qquad (31)$$

We assume the radial velocity fluctuation is given by the $\boldsymbol{E} \times \boldsymbol{B}$ drift velocity $\tilde{v} = -(c/B)\partial\phi/\partial y$. If we substitute the Fourier representations for f and ϕ into Eq. (31), the mixed mode terms may vanish due to the phase mixing, and the pairs of the same mode terms may survive, i.e., the flux is also decompose into three terms corresponding to the spectral decomposition:

$$\Gamma = \Gamma_0 + \Gamma_\mathrm{b} + \Gamma_\mathrm{c} . \qquad (32)$$

The discrete mode term Γ_0 is written as

$$\Gamma_0 = \frac{c}{B}\, e^{2\gamma t} \sum_k \frac{k_y k^2 \gamma^2}{4\omega_c^2 \pi e}\, \mathrm{Im}\chi(k, \omega) \left| \phi_\mathrm{s} \right|^2 , \qquad (33)$$

where ω_c is a characteristic frequency, ϕ_s is the shielded potential of the form of Eq. (21). The flux Γ_0 grows in time in unstable plasmas [5]. It may tends to a steady state when the growth rate γ is balanced with certain nonlinear damping rate Dk^2. Γ_0 is the same for ions and electrons, because $\mathrm{Im}\chi_\mathrm{i}$ = · $\mathrm{Im}\chi_\mathrm{e}$ from the dispersion relation (16), i.e., the anomalous transport due to the discrete mode (instability) is basically ambipolar. If the saturation level of the normalized potential is $|e\phi_\mathrm{s}/T|^2 = (L_n k)^{-2}$ [8], the flux can be approximated by $\Gamma_0 = -\gamma/k^2 N'$, i.e., the spatial diffusion coefficient $D = \gamma/k^2$ is recovered.

For the flux due to the ballistic mode, if we take the first term in Eq. (25), which is the direct beam mode from source, combined with Eq. (29), we have

$$\Gamma_{bi} = \sum_k \frac{ck_y}{e_i B} \frac{\pi}{k^2} \int \mathrm{d}^3 v \int \mathrm{d}^3 v' \,\delta(\omega_v - \omega_{v'}) \times \sum_j \frac{e_i \tilde{f}_i(k, v, 0)\, e_i \tilde{f}_i(-k, v', 0)}{\left| \varepsilon(k, \omega_v(k, v)) \right|^2 \omega_c} \mathrm{Im}\,\varepsilon(k, \omega_v(k, v)). \qquad (34)$$

For a single species case, expressing the shielded potential by

$$\phi_\mathrm{s} = -\frac{4\pi}{k^2} \int \mathrm{d}^3 v\, \frac{e\tilde{f}(k, v, 0)}{\omega_c\, \varepsilon(k, \omega_w)}$$

and separate $\mathrm{Im}\varepsilon(\mathrm{k}, \omega_v)$ from the v-integral, Eq. (34) may approximately be written by

$$\Gamma_\mathrm{b} = -\sum_k \frac{ck_y T}{eB} \left| \frac{e\phi_\mathrm{s}}{T} \right|^2 \int \mathrm{d}^3 v' J_0^2(\omega_v - \omega_\mathrm{T}^*) f_\mathrm{M}\, \pi\delta(\omega_v - \omega_{v'}).$$

The continuum contribution Γ_c can be derived in the similar way [4]. Since the continuum contributions need not satisfy the dispersion relation, Γ_b and Γ_c for ions and electron are, in general, different, i.e., the anomalous transports due to the continuum are not ambipolar.

4. Summary

The propagator for the Vlasov-Poisson system has been derived for an inhomogeneous plasma in a toroidal system, which has been applied for the spectral decomposition of the perturbed distribution and scalar potential. The distribution function has been decomposed into the three portions; the discrete mode determined from the dispersion relation, the ballistic mode, and the pure continuum contribution which represents the interaction between the ballistic mode and plasma cloud (clump) trapped by the shielded potential [9]. The flux due to the discrete mode is basically ambipolar. This does not hold for the flux due to the continuum contribution, which may cause the generation of the radial electric field.

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