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Analytical Global Model for Helical System

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Abstract

A rotating helical coordinate system is introduced to analyze magnetic systems with helical mirror symmetries. The parity nature of scalars and vectors is defined and conservation laws of parity are verified for vector calculus operations.

A global model of LHD type helical systems is derived based on this parity nature of vector field. Closed magnetic surfaces, islands, chaotic field line regions, and divertor field line regions are expressed comprehensively in this unified expression.

Keywords:

helical system, parity symmetry, LHD, analytical global model

1. Introduction

Symmetry plays an important role in physics. There are two types of symmetry; one is continuous symmetry and another is discrete symmetry. Axial symmetry is typical of the former one. Mirror reflecting symmetry and finite angle rotating symmetry are typical examples of the latter symmetry.In quantum mechanics, corresponding quantum numbers are present for both (continuous and discrete symmetries). Especially, the quantum number for the mirror reflection symmetry is called parity. The parity determines the selection rule for the transition processes.

In classical mechanics, continuous symmetry leads to the conservation laws but discrete symmetry does not do so directly. Nevertheless, discrete symmetry should have an important role for plasma confinement, because the chaotic region in the phase space of a particle motion will be very much reduced due to the limitation of movable space if the Hamiltonian is discretely symmetric.

We can analyze the tokamak theoretically with the assumption of continuous symmetry (axial symmetry). On the other hand, helical systems can possess rigorous discrete symmetry (periodicity and helical mirror symmetry). In the present paper, we develop an analytical treatment of helical systems based on the discrete symmetry.

The results of numerical computations for high energy particle orbits in helical systems depend on the assumption of the loss boundary. The loss boundary for particle orbits is conventionally assumed to be the outermost magnetic surface. In this case, a large part of the reflecting particles are lost in a very short time. Then, a big loss cone appears in the particle phase space. On the other hand, if the loss boundary of particle orbits is placed on the vacuum vessel wall, almost all reflecting particles can have clear drift surfaces if the starting positions of particle motion are placed inside the magnetic surfaces. If the pitch angle is placed between the passing particle and reflecting particle, the particle orbits show a chaotic nature and are eventually lost to vacuum vessel wall. Even in this case, the life time is very long compared to the transit time (L/v_{th}: L is distance to the vacuum vessel wall, v_{th} is thermal velocity of particle). The loss cone should disappear in this case[1].

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Then a global model is necessary to estimate the performance of a helical magnetic field for plasma confinement. Closed magnetic surface regions, islands, chaotic field line regions and divertor field line regions should be represented by unified expression.

In section 2, we introduce a rotating helical coordinate system which is appropriate to handle the helical mirror symmetry. Parity is defined for vectors and scalars. The parity conservation laws for vector calculus operations are verified. In section 3, we point out that the MHD equilibrium of helical systems can be expressed using even vectors $\boldsymbol{B}$, $\boldsymbol{J}$ and even scalar P. An explicit analytical expression for the magnetic field is derived. Section 4 is devoted to conclusions.

2. Parity Conservation Laws for Vectors

It is important to use an appropriate coordinate system for the analysis of symmetry. For LHD type helical systems, the rotating helical coordinate system (X, Y, ϕ) is an appropriate one. It is defined by the relations

$$\boldsymbol{r} \equiv (x, y, z)^{\mathrm{T}} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = X\boldsymbol{i} + Y\boldsymbol{j} - r_0 \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}\phi} \qquad (1)$$

where

$$\boldsymbol{i} = \begin{pmatrix} \cos\theta \sin\phi \\ \cos\theta \cos\phi \\ \sin\theta \end{pmatrix}, \quad \frac{\mathrm{d}\boldsymbol{i}}{\mathrm{d}\phi} = \cos\theta \boldsymbol{k} + p\boldsymbol{j}, \qquad (2)$$

$$\boldsymbol{j} = \begin{pmatrix} -\sin\theta \sin\phi \\ -\sin\theta \cos\phi \\ \cos\theta \end{pmatrix}, \quad \frac{\mathrm{d}\boldsymbol{j}}{\mathrm{d}\phi} = -\sin\theta \boldsymbol{k} - p\boldsymbol{i}, \qquad (3)$$

$$\boldsymbol{k} = \begin{pmatrix} \cos\phi \\ -\sin\phi \\ 0 \end{pmatrix}, \quad \frac{\mathrm{d}\boldsymbol{k}}{\mathrm{d}\phi} = -\cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}, \qquad (4)$$

$$\frac{\mathrm{d}^2\boldsymbol{k}}{\mathrm{d}\phi^2} = -\boldsymbol{k}. \qquad (5)$$

r_0 is the major radius of the device (=3.9 m for LHD), p is pitch number of the helical winding (=5 for LHD) and ϕ is the conventional toroidal angle ($\theta \equiv p\phi$). $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are a set of right handed rectangular unit vectors.

$$\boldsymbol{i} \times \boldsymbol{j} = \boldsymbol{k}, \quad \boldsymbol{j} \times \boldsymbol{k} = \boldsymbol{i}, \quad \boldsymbol{k} \times \boldsymbol{i} = \boldsymbol{j} \qquad (6)$$

Any vector $\boldsymbol{A}$ can be represented as follows.

$$\boldsymbol{A} = A_1 \boldsymbol{i} + A_2 \boldsymbol{j} + A_3 \boldsymbol{k},$$
$$A_1 = \boldsymbol{A} \cdot \boldsymbol{i}, \quad A_2 = \boldsymbol{A} \cdot \boldsymbol{j}, \quad A_3 = \boldsymbol{A} \cdot \boldsymbol{k}. \qquad (7)$$

The vector calculus operations [2] are reduced to

$$\nabla f = \boldsymbol{i}\frac{\partial f}{\partial X} + \boldsymbol{j}\frac{\partial f}{\partial Y} + \boldsymbol{k}\left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{pY}{r}\frac{\partial}{\partial X} - \frac{pX}{r}\frac{\partial}{\partial Y}\right) f, \qquad (8)$$

$$\nabla \cdot \boldsymbol{A} = \left(\frac{\partial}{\partial X} + \frac{\cos\theta}{r}\right) A_1 + \left(\frac{\partial}{\partial Y} - \frac{\sin\theta}{r}\right) A_2 + \left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{pY}{r}\frac{\partial}{\partial X} - \frac{pX}{r}\frac{\partial}{\partial Y}\right) A_3, \qquad (9)$$

$$\begin{aligned}\nabla \times \boldsymbol{A} = \boldsymbol{i}&\left[\left(\frac{\partial}{\partial Y} - \frac{\sin\theta}{r}\right) A_3 - \left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{pY}{r}\frac{\partial}{\partial X} - \frac{pX}{r}\frac{\partial}{\partial Y}\right) A_2 - \frac{p}{r} a_1\right] \\ + \boldsymbol{j}&\left[\left(\frac{1}{r}\frac{\partial}{\partial\phi} + \frac{pY}{r}\frac{\partial}{\partial X} - \frac{pX}{r}\frac{\partial}{\partial Y}\right) A_1 - \left(\frac{\partial}{\partial X} + \frac{\cos\theta}{r}\right) A_3 - \frac{p}{r} A_2\right] \\ + \boldsymbol{k}&\left[\frac{\partial A_2}{\partial X} - \frac{\partial A_1}{\partial Y}\right],\end{aligned} \qquad (10)$$

where r is radial distance in a conventional cylindrical coordinate system,

$$r \equiv r_0 + X\cos(p\phi) - Y\sin(p\phi). \qquad (11)$$

In the following, we restrict the discussions only to vectors and scalars whose ϕ dependence are expressible only through functions of $\cos\theta$ and $\sin\theta$, for example,

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} B_1(X, \cos\theta, Y, \sin\theta) \\ B_2(X, \cos\theta, Y, \sin\theta) \\ B_3(X, \cos\theta, Y, \sin\theta) \end{pmatrix}. \qquad (12)$$

Helical mirror symmetry at $\phi = 0$ is given by the relation of magnetic field $\boldsymbol{B}(X, Y, \phi)$ and $\mathbf{B}(X, -Y, -\phi)$. Helical mirror symmetry at $\phi = \pi/2p$ is given by the relation of magnetic field $\boldsymbol{B}(X, Y, \phi)$ and $\boldsymbol{B}(-X, Y, \pi/p-\phi)$. For an LHD type magnetic field, the following symmetric relations are possible.

$$\begin{aligned}B_1(X, -Y, -\phi) &= B_1(X, \cos\theta, -Y, -\sin\theta) \\ &= -B_1(X, \cos\theta, Y, \sin\theta) = -B_1(X, Y, \phi),\end{aligned}$$

$$\begin{aligned}B_2(X, -Y, -\phi) &= B_2(X, \cos\theta, -Y, -\sin\theta) \\ &= B_2(X, \cos\theta, Y, \sin\theta) = B_2(X, Y, \phi),\end{aligned}$$

$$\begin{aligned}B_3(X, -Y, -\phi) &= B_3(X, \cos\theta, -Y, -\sin\theta) \\ &= B_3(X, \cos\theta, Y, \sin\theta) = B_3(X, Y, \phi),\end{aligned}$$

$$\begin{aligned}B_1(-X, Y, \pi/p-\phi) &= B_1(-X, -\cos\theta, Y, \sin\theta) \\ &= B_1(X, \cos\theta, Y, \sin\theta) = B_1(X, Y, \phi),\end{aligned}$$

$$\begin{aligned}B_2(-X, Y, \pi/p-\phi) &= B_2(-X, -\cos\theta, Y, \sin\theta) \\ &= -B_2(X, \cos\theta, Y, \sin\theta) = -B_2(X, Y, \phi),\end{aligned}$$

$$\begin{aligned}B_3(-X, Y, \pi/p-\phi) &= B_3(-X, -\cos\theta, Y, \sin\theta) \\ &= B_3(X, \cos\theta, Y, \sin\theta) = B_3(X, Y, \phi),\end{aligned}$$

EQUILIBLIUM OF HELICAL PLASMA

$$B = \nabla \times A$$
$$J = \nabla \times B / \mu_0$$
$$\nabla P = J \times B$$

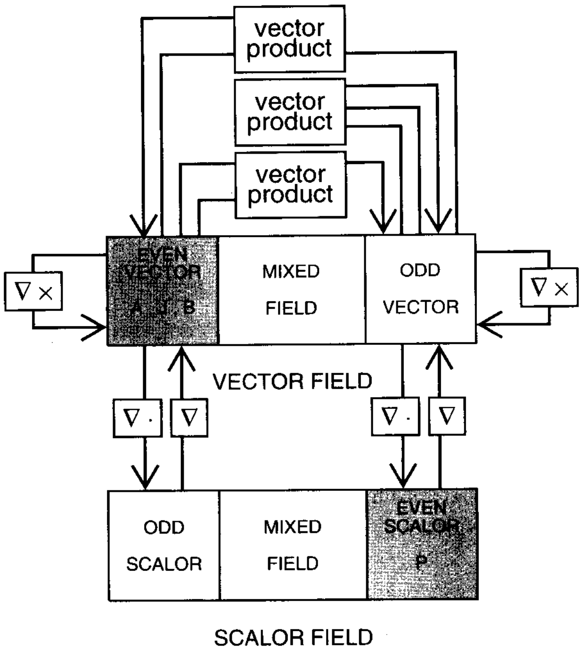

Fig. 1 Parity conservation laws for helically symmetric field.

Then we define even parity vectors as having the following parity nature,

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix} = \begin{pmatrix} \Sigma a_e(X,\cos(p\phi)) \times b_o(Y,\sin(p\phi)) \\ \Sigma c_o(X,\cos(p\phi)) \times d_e(Y,\sin(p\phi)) \\ \Sigma e_e(X,\cos(p\phi)) \times f_e(Y,\sin(p\phi)) \end{pmatrix} \equiv \begin{pmatrix} [eo] \\ [oe] \\ [ee] \end{pmatrix}, \tag{13}$$

where the subscript e(o) represents even (odd) functions of each argument. The symbol $[eo]$ represents that the function is a summation of products of even and odd functions with (X, $\cos\theta$) and (Y, $\sin\theta$). Odd parity vector is defined by

$$\begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}_{\text{odd}} = \begin{pmatrix} [o\,e] \\ [e\,o] \\ [o\,o] \end{pmatrix}. \tag{14}$$

Even scalar f_{even} and odd scalar f_{odd} are also defined in a similar way.

$$f_{\text{even}} = [e\,e]\,, \tag{15}$$

$$f_{\text{odd}} = [o\,o]\,. \tag{16}$$

For these scalars and vectors, the following relationships (parity conservation law for helically symmetric field) can be verified directly.

$$\nabla(\text{even(odd) scalar}) = \text{odd(even) vector}\,, \tag{17}$$

$$\nabla \cdot (\text{even(odd) vector}) = \text{odd(even) scalar}\,, \tag{18}$$

$$\nabla \times (\text{even(odd) vector}) = \text{even(odd) vector}\,, \tag{19}$$

$$(\text{even vector}) \times (\text{even vector}) = \text{odd vector}\,, \tag{20}$$

$$(\text{odd vector}) \times (\text{odd vector}) = \text{odd vector}\,, \tag{21}$$

$$(\text{even vector}) \times (\text{odd vector}) = \text{even vector}\,, \tag{22}$$

$$(\text{odd vector}) \times (\text{even vector}) = \text{even vector}\,. \tag{23}$$

These relations are summarized in Fig.1.

3. MHD equilibrium of Helical Systems

Parity conservation laws for helically symmetric fields simplify the relations of MHD equilibrium,

$$\nabla \times \boldsymbol{A} = \boldsymbol{B}, \tag{24}$$

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \tag{25}$$

$$\nabla P = \boldsymbol{J} \times \boldsymbol{B}. \tag{26}$$

Vector potential $\boldsymbol{A}$, magnetic field $\boldsymbol{B}$, and current density $\boldsymbol{J}$ should have the same parity nature (even parity vectors), because they are related by rotation ($\nabla\times$). Then the pressure distribution P should be an even scalar.

For example, the vacuum magnetic field for LHD type helical systems is obtained as follows.First we write down a general expression for even vectors for the magnetic potential $\boldsymbol{A}$.

$$\boldsymbol{A} = \frac{B_p}{a} \begin{pmatrix} \sin\theta(X^2 a_1 + Y^2 a_2) + \cos\theta XY a_3 \\ \cos\theta(X^2 b_1 + Y^2 b_2) + \sin\theta XY b_3 \\ X^2 c_1 + Y^2 c_2 + \sin 2\theta XY c_3 + \cos 2\theta(X^2 c_4 + Y^2 c_5) \end{pmatrix} \tag{27}$$

The coefficients B_p, a, $a_1 \sim c_5$ are determined by the equation

$$0 = \nabla \times (\nabla \times \boldsymbol{A}). \tag{28}$$

The lowest solution reduces to

$$\boldsymbol{A}_1 = \frac{B_p}{a} \begin{pmatrix} 0 \\ 0 \\ (Y^2 - X^2)/2 \end{pmatrix}. \tag{29}$$

If we include more higher order terms, we get

$$A = \frac{B_p}{a} \begin{pmatrix} -\frac{p}{3r} Y^3 - \frac{p^3}{12r^3} Y(X^4 + Y^4) \\ -\frac{p}{3r} X^3 - \frac{p^3}{12r^3} X(X^4 + Y^4) \\ -\frac{X^2 - Y^2}{2}\left(1 - \frac{X\cos\theta - Y\sin\theta}{2r} - \frac{p^4}{3r^4} X^2 Y^2\right) \end{pmatrix}. \tag{30}$$

Then, an analytical expression for the magnetic field of LHD is given by

$$B = B_0 \begin{pmatrix} 0 \\ 0 \\ \frac{r_0}{r} \end{pmatrix} + \frac{B_p}{a} \begin{pmatrix} Y + \frac{p^2 Y(3X^2 + Y^2)}{3r^2} + \frac{p^4 Y(5X^4 - 6X^2Y^2 + Y^4)}{6r^4} + \frac{p^4 X^2 Y^2 (X^2 - Y^2)}{2r^5} \sin\theta + \frac{(X^2 + Y^2)\sin\theta - 2XY\cos\theta}{4r} \\ X + \frac{p^2 X(X^2 + 3Y^2)}{3r^2} + \frac{p^4 X(X^4 - 6X^2Y^2 + 5Y^4)}{6r^4} + \frac{p^4 X^2 Y^2 (X^2 - Y^2)}{2r^5} \cos\theta - \frac{(X^2 + Y^2)\cos\theta - 2XY\sin\theta}{4r} \\ -\frac{p(X^2 - Y^2)}{r} - \frac{p^3(X^4 - Y^4)}{3r^3} + \frac{p^3(X^4 + Y^4)(X\cos\theta + Y\sin\theta)}{4r^4} + \frac{p(X^3\cos\theta + Y^3\sin\theta)}{3r^2} \end{pmatrix} \tag{31}$$

Poincaré plots of this magnetic field reconstruct closed magnetic surfaces, islands, chaotic field line regions, and divertor field lineregions very well, showing the structure of the actual LHD vacuum field [2].

4. Summary

Besides the M-fold periodicity in the toroidal angle, helical mirror symmetry can exist strictly in helical systems. These discrete type symmetries are treated by using a rotating helical coordinate system. The parity nature of scalars and vectors is introduced and parity conservation laws are derived.

Global structures of helical systems are analyzed based on these discrete type symmetry relations. Vacuum magnetic field is shown analytically which can express comprehensively closed, chaotic and divertor field line regions.

If finite conductivity is introduced, the rigorous helical mirror symmetry relation for equilibrium is modified and diffusion flux of plasma will take place.

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