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Discrete and Continuum Ballooning Modes in a Stellarator

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Abstract

By studying the α, ψ, θ_k dependence of ballooning mode growth rates in a Mercier-unstable equilibrium case modelling the Large Helical Device (LHD) with a broad pressure profile, it is found that two distinct topological types coexist – a topologically cylindrical branch and a topologically spherical branch. It is shown that the "cylindrical" branch can be described by a ripple-expansion of the ballooning equation, carried beyond lowest order in the number of field periods. However the "spherical" branch cannot be found from such an expansion at any finite order. According to WKB theory, the cylindrical and spherical branches give rise to quasi-discrete modes and continuum global modes, respectively. The cylindrical branch disappears for the Mercier-stable peaked-pressure-profile LHD cases and can thus be regarded as a finite-growth-rate interchange mode.

Keywords:

MHD, LHD, WKB, ballooning, stability, spectrum, three-dimensional geometry

The application of the 3D-WKB ballooning formalism[1] to a realistic stellarator equilibrium is a computationally intensive task, and the formalism has only recently been applied[2]. The case studied (β=4 % at magnetic axis in Table III of Ref.[3]) was essentially LHD, but with a pressure profile chosen to be broader than the one intended for normal operation. This choice makes the equilibrium strongly Mercier unstable[4], thus ensuring that a rich spectrum of unstable modes of low-toroidal mode number is available for numerical study. It was found[2] that the WKB formalism[1] predicted a global (quasi-)discrete spectrum in good agreement with that calculated by the global eigenvalue code TERPSICHORE, after painstaking convergence studies were done for TERPSICHORE.

Nakajima[4] has also studied a case with a narrower pressure profile that is only very weakly Mercier unstable at 4 % β on axis, and enters a second region of Mercier stability at higher β values. However, these interchange-stable equilibria are found to be unstable to a strongly localized ballooning mode. Since the marginal stability surface for this instability is topologically spherical in the reduced ray phase space, this mode belongs to the "broad unstable continuum"[1].

In this paper we show that the 4 %-β, Mercier unstable "broad pressure profile case" used in the previous study[2] is *also* unstable to the continuum ballooning mode of Nakajima. Thus we have found a

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physical case where a broad unstable spectrum and a quasi-discrete spectrum coexist.

We label magnetic surfaces variously by the parameter s used in the VMEC equilibrium code, by the enclosed poloidal flux $2\pi\psi$, or by the inverse rotational transform $q \equiv 1/\iota$ [assuming $q(s)$ to be monotonic – in the LHD case studied, $q(0)=2.3$, $q(1)=0.88$]. Introducing straight-field-line poloidal and toroidal angles θ and ζ, and a Clebsch potential α, constant on a magnetic field line, $\alpha \equiv \zeta - q\theta$, we have $\boldsymbol{B}=\nabla\alpha\times\nabla\psi$.

We change the coordinates on a magnetic surface from ζ, θ to α, θ taking the domain to be the plane [1]. We assume the perpendicular-displacement stream function to be given by $\tilde{\varphi}=\varphi\exp(iS/\epsilon - i\omega t)$, where $\varphi(\theta, \psi, \alpha)$ is assumed to vary on the equilibrium scale, whereas the phase variation is taken to be rapid. The frequency $\omega = i\gamma$, where γ is the growth rate, is assumed $O(1)$, which requires that $S=S(\alpha, \psi)$. From the definition of the wave vector, $\boldsymbol{k}=k_\alpha\nabla\alpha+k_q\nabla q \equiv k_\alpha(\nabla\alpha+\theta_k\nabla q)$ where $k_\alpha \equiv \partial S/\partial\alpha$ and $k_q \equiv \partial S/\partial q$.

Here the ballooning parameter θ_k appears as the ratio k_q/k_α. This parameter is angle-like in the sense that the periodicity properties of the ballooning growth rate with respect to θ_k are the same as those of equilibrium quantities with respect to θ. *i.e.* the growth rate is invariant under the operations T: $\alpha \to \alpha + 2\pi$, $\theta_k \to \theta_k$ and P: $\alpha \to \alpha - 2\pi q(\psi)$, $\theta_k \to \theta_k + 2\pi$. This skewed symmetry, combined with magnetic shear, is seen in Fig. 1, which was produced by solving the

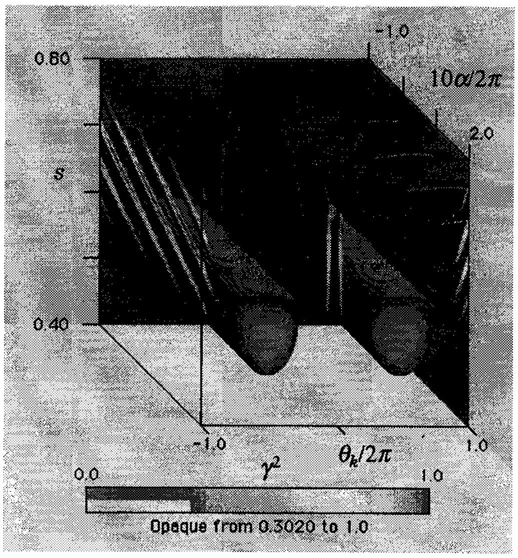

Fig. 1 Isosurfaces of ballooning growth rate in the three-dimensional phase space.

ballooning eigenvalue equation for parameters on a three-dimensional grid in θ_k, α, s space.

The eigenvalue equation is

$$\frac{\partial}{\partial\theta}\mathscr{A}\frac{\partial\varphi}{\partial\theta} - (\mathscr{K} - \mathscr{N}\omega^2)\,\varphi = 0\,, \tag{1}$$

where $\mathscr{A} \equiv |\boldsymbol{k}|^2/k_\alpha^2\,\mathscr{J}B^2$, $\mathscr{K} \equiv -2p'(\psi)\mathscr{J}\,\kappa\cdot\boldsymbol{B}\times\boldsymbol{k}/B^2k_\alpha$, and $\mathscr{N}$ is an inertia factor. Here $\mathscr{J} \equiv 1/\boldsymbol{B}\cdot\nabla\theta$ is the Jacobian factor, p is the pressure and κ the field-line curvature.

Both types of mode we wish to study can be studied using the ballooning equation, but have the following distinguishing features (for intermediate growth rates): (1) the *interchange branch* has growth-rate isosurfaces that are "topologically cylindrical", whereas the *ballooning modes* have "topologically spherical" isosurfaces; (2) the eigenfunctions for the ballooning branch are localized to a few field periods, whereas the eigenfunctions for the interchange branch are more extended; (3) the local growth rate for the ballooning branch is at its maximum near $\theta_k=0$, whereas the local growth rate for the interchange branch peaks around $\theta_k=\pi$.

To understand the nature of the interchange modes we make an expansion in the inverse of the number of field periods, M, to derive a ripple-averaged ballooning equation that contains the interchange branch but eliminates the ballooning branch.

The coefficients $\mathscr{A}$, $\mathscr{K}$ and $\mathscr{N}$ have a slow dependence on θ, including "secular terms" in $\theta - \theta_k$, and a 2π-periodic dependence on the rapid helical variable $u \equiv M\zeta - l\theta$, where $M=10$, $l=2$ in LHD. We thus use a two-scale approach in which the derivative along the field lines becomes

$$\frac{\mathrm{d}}{\mathrm{d}\theta} = \frac{\partial}{\partial\theta}(Mq - l)\frac{\partial}{\partial u}\,. \tag{2}$$

We expand φ in inverse powers of $Mq - l$. From the solubility condition for $\varphi^{(2)}$ at $O(M^0)$ we find the *ripple-averaged ballooning equation*

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\left\langle\frac{1}{\mathscr{A}}\right\rangle^{-1}\frac{\mathrm{d}\varphi^{(0)}}{\mathrm{d}\theta}\right) - (\langle\mathscr{K}\rangle - \langle\mathscr{N}\rangle\,\omega^2)\,\varphi^{(0)}=0\,, \tag{3}$$

where $\langle\cdot\rangle$ denotes an average over one period in u, leaving only θ dependence, as in the axisymmetric case.

When the TERPSICHORE normalization factor [2] was used for $\mathscr{N}$, the leading order ripple-averaged approximation given by Eq. 3 reproduced the behaviour of the interchange branch in a qualitatively correct fashion (i.e. being maximum at $\theta_k=\pi$), but if the more usual incompressible MHD $\mathscr{N}$ was used it was

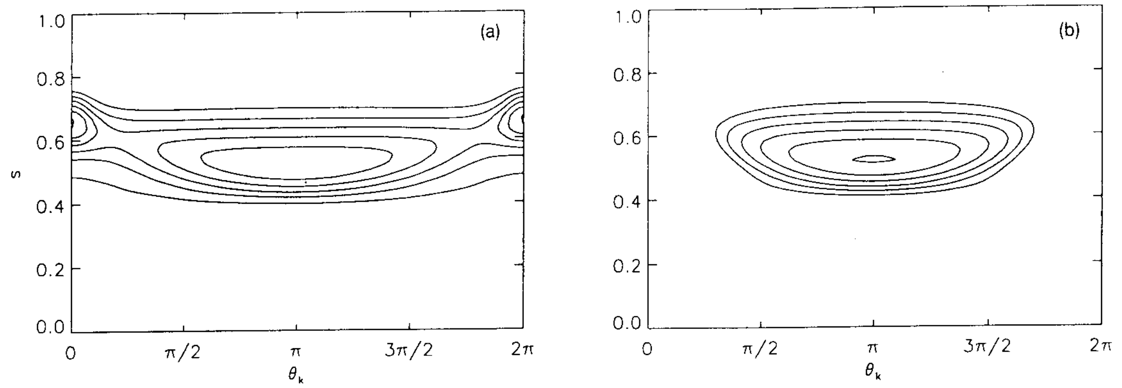

Fig. 2 Contours of ballooning eigenvalue ω^2 calculated (a) by solving Eq. (1), then averaging over α, and (b) using Eq. (3), plus the next order correction.

found that the ripple expansion had to be carried to next order (see Fig. 2). Since this expansion averages out all α dependence it can never reproduce the ballooning branch, which is thus inherently three-dimensional. We conjecture that the singular nature of the continuum eigenfunction may make the ballooning branch less dangerous for anomalous transport.

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References

[1] R.L. Dewar and A.H. Glasser, Phys. Fluids **26**, 3038 (1983).

[2] W.A. Cooper, D.B. Singleton and R.L. Dewar, Phys. Plasmas **3**, 275 (1996), erratum: Phys. Plasmas **3**, 3520 (1996).

[3] Y. Nakamura *et al.*, J. Comput. Phys. **128**, 43 (1996).

[4] N. Nakajima, Phys. Plasmas **3**, 4556 (1996).