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Ion acoustic soliton with finite vorticity
有限の渦を持つイオン音波ソリトン

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We show that the series of Kadomtsev-Petviashvili equations (including higher order series) describe only "zero-vorticity" motion of ion acoustic waves. We derive a new system of equations that has a finite vorticity.

1. Introduction

While there are a lot of works on the ion acoustic solitons, they have not paid attention to the "vortex" of the wave field. In fact, the Korteweg-de Vries (KdV) equation [1] considers a spatially one-dimensional system, so the fields do not have a freedom to create a vorticity.

The two-dimensional (or three-dimensional) generalization of the KdV equation is the Kadomtsev-Petviashvili (KP) equation [2, 3], which, however, is set to eliminate vorticity; in the present work we show that the series of higher order KP equations are all vortex free.

This observation motivates us to elucidate how the vortex-free solitons are special in a wider phase space, and to formulate and analyze a generalized system that has a finite vorticity.

2. Hamiltonian system and Casimir invariant

A generalized Hamiltonian system [4] is written as

$$\partial_t u = \mathcal{J}(u)\partial_u H(u), \tag{1}$$

where u is the state vector, $\mathcal{J}$ is the Poisson operator, H is the Hamiltonian, and ∂_u is the functional derivative. Poisson bracket is defined as $\{F, G\} = \int (\partial_u F)(\mathcal{J}\partial_u G)\,\mathrm{d}x$, and the time evolution of a functional $F(u)$ is written as $\partial_t F = \{F, H\}$.

When Poisson operator has non-trivial kernel, the system is called non-canonical. In a non-canonical system, a functional $C(u)$ such that $\mathcal{J}\partial_u C = 0$ is conserved, and called Casimir invariant. A Casimir invariant foliates the phase space, hence the dynamics is constrained on the "Casimir leaf".

Ion acoustic waves are governed by the continuity equation, the equation of motion, and Poisson equation:

$$\partial_t n + \nabla\cdot(n\boldsymbol{u}) = 0, \tag{2a}$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\phi = 0, \tag{2b}$$

$$n = \mathrm{e}^{\phi} - \Delta\phi. \tag{2c}$$

Here, n is the number density, $\boldsymbol{u}$ is the velocity, and ϕ is the electrostatic potential (all variables are normalized). These equations are written in generalized Hamiltonian form (1) with $u = (n, \boldsymbol{u})^\top$ and following Poisson operator and Hamiltonian [5]:

$$\mathcal{J} = \begin{pmatrix} 0 & -\nabla\cdot \\ -\nabla & -n^{-1}(\nabla\times\boldsymbol{u})\times \end{pmatrix}, \tag{3}$$

$$H = \int \left(\frac{1}{2}n|\boldsymbol{u}|^2 + \mathcal{N}(n)\right)\mathrm{d}x, \tag{4}$$

where $\mathcal{N}(n) = \mathrm{e}^{\phi}(\phi - 1) + (\nabla\phi)^2/2 + 1$.

This system is non-canonical, and two Casimir invariants are known. One is total number, and the other is expressed with vorticity $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$. If the spatial dimension is three, the helicity $\int \boldsymbol{u}\cdot\boldsymbol{\omega}\,\mathrm{d}x$ is Casimir invariant; and if the spatial dimension is two, the generalized enstrophy $\int nf(\omega/n)\,\mathrm{d}x$ is Casimir invariant (f is an arbitrary function).

Next, suppose that vorticity is zero. This condition makes above Casimir invariants zero (trivialized). Furthermore, the structure of Poisson operator (3) is changed, and the functional $\int \boldsymbol{u}\cdot\boldsymbol{c}\,\mathrm{d}x$ ($\boldsymbol{c}$ is an arbitrary constant vector) becomes Casimir invariant.

In this way, fluids with zero-vorticity constrained on "singular" Casimir leaf [6]. However, most of studies on soliton have not focused on this point. In this work, generalization of the KP equation to finite vorticity is considered.

3. KP equation

At first, we see how vorticity vanishes in the KP equation. KP equation is derived by reductive perturbation method. Introducing a small parameter ϵ, dependent variables are expanded as $n = 1 + \epsilon^2 n_1 + \epsilon^4 n_2 + \cdots$, $u_x = \epsilon^2 u_1 + \epsilon^4 u_2 + \cdots$,

$u_y = \epsilon^2 v_1 + \epsilon^4 v_2 + \cdots$; and independent variables as $\xi = \epsilon(x - t)$, $\eta = \epsilon^2 y$, $\tau = \epsilon^3 t$. Then, we obtain $n_1 = u_1 = \phi_1$ from ϵ^3-order terms of equation (2), $\partial_\xi v_1 = \partial_\eta \phi_1$ from ϵ^4-order, and the KP equation from ϵ^5-order:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi}{\partial \xi^3}\right) + \frac{1}{2}\frac{\partial^2 \phi}{\partial \eta^2} = 0, \quad (5)$$

where ϕ_1 is simply written as ϕ.

The dynamics governed by the KP equation has no vorticity: $\omega_1 = \partial_\xi v_1 - \partial_\eta u_1 = \partial_\eta \phi_1 - \partial_\eta \phi_1 = 0$. Furthermore, higher order vorticity $\omega_2 = \partial_\xi v_2 - \partial_\eta u_2$, can be calculated from ϵ^6-order terms of equation (2), becomes zero. Similarly, all order vorticities become zero. While the KP equation is spatially two-dimensional, generalization to three-dimension is easy. Variables are expanded as $\zeta = \epsilon^2 z$, $u_z = \epsilon^3 w_1 + \epsilon^5 w_2 + \cdots$, and the last term of equation (5) is changed as $\partial_\eta^2 \to \Delta_\perp := \partial_\eta^2 + \partial_\zeta^2$. In the three-dimensional case, however, all vorticities vanish too. Since the basic equations (2) can have finite vorticity, we should see how the reductive perturbation method kills vorticity.

First of all, orders of vorticity and compressibility $\nabla \cdot \boldsymbol{u}$ should be compared. Leading order of vorticity $(\partial_\xi v_1 - \partial_\eta u_1)$ is ϵ^4, and that of compressibility $(\partial_\xi u_1)$ is ϵ^3. This is because the reductive perturbation method makes compressibility and dispersion balance, and soliton appears [7].

In addition, we see the effect of vorticity in the equation of motion (2b). The convection term is rewritten as $(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\boldsymbol{u} \times \boldsymbol{\omega} + \nabla(|\boldsymbol{u}|^2/2)$, so the effect of vorticity appears only in the first term. The leading order of this term $(u_1 \omega_1)$ is ϵ^6, hence the effect has higher order than the KP equation. Thus, we can say that the ordering pushes the effect of vorticity higher (smaller) than the governing equation.

4. Generalization to a finite vorticity system

Now we introduce finite vorticity in the KP equation. From the previous section, we can say that the problem is that the order of vorticity is low. Since a soliton appears when compressibility and dispersion are balanced, we should not make the order of vorticity higher than that of compressibility. Thus the order of vorticity should be ϵ^3. With this order, the vorticity effect $\boldsymbol{u} \times \boldsymbol{\omega}$ has ϵ^5-order, same as the KP equation.

ϵ^3-order vorticity is realized by adding ϵ^1-order velocities v_0 and w_0 to u_y and u_z. For simplicity, we suppose $v_0 = \partial_\zeta \psi$, $w_0 = -\partial_\eta \psi$, and $\partial_\xi \psi = 0$. From the ϵ^4-order of equation (2b), we obtain the equations of motion of v_0 and w_0. They are equivalent to two-dimensional Euler vorticity equation:

$$\frac{\partial}{\partial \tau}(\Delta_\perp \psi) + [\Delta_\perp \psi, \psi] = 0, \quad (6)$$

where $[f, g] = \partial_\eta f \partial_\zeta g - \partial_\zeta f \partial_\eta g$. From the ϵ^5-order of equation (2), we obtain modified three-dimensional KP equation:

$$\frac{\partial}{\partial \xi}\left(\frac{\partial \phi}{\partial \tau} + \phi \frac{\partial \phi}{\partial \xi} + \frac{1}{2}\frac{\partial^3 \phi}{\partial \xi^3} + [\phi, \psi]\right) + \frac{1}{2}\Delta_\perp^2 \phi = 0. \quad (7)$$

We call the set of equations (6) and (7) Kadomtsev-Petviashvili-Yoshida (KPY) equations.

5. Summary

It is showed that there are no vorticity in the (two- and three-dimensional) KP equation, hence the dynamics is constrained on singular Casimir leaf. In order to put the dynamics on non-singular (vortex) Casimir leaf, the new (lower) order velocities are introduced. The resulting equations are the KPY equations (6)–(7), the first is the Euler vorticity equation and the second is the modified KP equation. Vorticity, compressibility, and dispersion are balanced in this system. Results of numerical and analytical studies will be presented in the conference.

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