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A Stochastic Model of Inward Diffusion in Magnetospheric Plasmas

Naoki Sato, Zensho Yoshida and Yohei Kawazura

Graduate School of Frontier Sciences, The University of Tokyo, Kashiwa, Chiba 277-8561, Japan

A model is developed to explain the *inward diffusion* of plasmas in dipole magnetic fields, which seemingly contradicts the entropy principle by creating “gradients”. We consider a diffusion process on a *proper frame* that models the macroscopic hierarchy of magnetized particles. Transforming back to the laboratory frame the Fickian diffusion on the proper frame, we observe creation of gradients. Theory is verified by numerical simulations of the obtained Fokker-Planck equation. We obtain a peaked density profile that mimics radiation belts in planetary magnetospheres.

1. Introduction

Empirical models have been developed to describe the density profiles observed in planetary magnetospheres [1-2]. The purpose of this research is to formulate a theory of inward diffusion [3-4] in a properly defined magnetic space that explains how the geometry of the magnetic field affects the considered transport mechanism. The key idea is the use of the foliated phase space modeling a macroscopic hierarchy [5]. Using the change of variable formula of stochastic analysis (Ito’s Lemma [6,7]), a set of stochastic differential equations (SDEs) and the associated Fokker-Planck equation (FPE) are obtained on the “proper frame” spanning the effective phase space of the foliation. The stochastic source is the macroscopic (compared to the characteristic scale of cyclotron motion) turbulent electric potential $\delta\varphi$.

2. $\boldsymbol{E}\times\boldsymbol{B}$ Driven Transport

To explain transport occurring in the direction $x_\perp$ normal to flux surfaces, violation of the third adiabatic invariant (the magnetic flux ψ) is invoked [1-2]. Electromagnetic fluctuations $\delta\boldsymbol{E}=-\nabla\delta\varphi$ generate a drift velocity

$$\dot{x}_\perp=\frac{\delta E_\perp}{B}=-\frac{1}{rB}\frac{\partial\delta\varphi}{\partial\vartheta} \tag{1}$$

Where $\boldsymbol{B}=\nabla\psi\times\nabla\vartheta$ is the dipole magnetic field and (r,z,ϑ) a cylindrical reference system (note that dipoles are ϑ-symmetric). If we express the turbulent potential $\delta\varphi$ with Gaussian white noise $\Gamma=\dot{W}$, equation (1) takes the form of an SDE (random variables are symbolized by upper-case letters. W indicates a Wiener process and $D_\perp^{1/2}$ is a measure of the amplitude of the fluctuations):

$$dX_\perp=D_\perp^{1/2}\frac{dW}{rB} \tag{2}$$

3. Definition of the Stochastic Integral

Due to the nowhere differentiability of the Wiener process, definition of the stochastic integral shows arbitrariness with respect to a parameter $\alpha\in[0,1]$ which physically represents the way W evolves in time. Depending on α, the solution to equation (2) changes. The case $\alpha=0$ is known as Ito’s definition, the choice $\alpha=1/2$ is due to Stratonovich [6,7]. This arbitrariness has two main consequences: the change of frame formula for the equations of motion depends on α and the FPE corresponding to the equations of motion is itself determined by α [6,7]. The change of variable formula for an arbitrary α cannot be found in the literature and it is derived in this work. By physical arguments we then show that the condition $\alpha=0$ is consistent with the physics of inward diffusion.

4. Change of Frame

To interpret inward diffusion as a flattening (Fickian) diffusion process, we need to find a new reference system $y(x_\perp,l,t)$ that satisfies the property (l is the length along field lines, t the time variable, and v_α a drift in the new frame):

$$dY=v_\alpha dt+D_\perp^{1/2}dW \tag{3}$$

In other words, the Wiener process W has to appear in the equation of motion as an independent factor. We further require $\partial_t y=\partial_l y=0$. On the other hand, y must satisfy the change of variable formula. This leads us to conclude that $y=\psi$ with

$$d\Psi=D_\perp\left(\frac{1}{2}-\alpha\right)\left[\frac{\nabla l\cdot\nabla\psi}{(rB)^2}\frac{\partial}{\partial l}+\frac{\partial}{\partial\psi}\right]ln(rB)dt+D_\perp^{1/2}dW \tag{4}$$

5. Dynamics along the Magnetic Field

Particles move along field lines and the corresponding parallel equations of motion are required. The parallel velocity $v_\parallel$ is determined by bounce motion, dynamical friction, and parallel turbulent electric field. Furthermore, since $\nabla\psi$ and ∇l are not normal vectors, the perpendicular motion affects the value of l. In summary[1]:

$$\begin{cases} dL = v_\parallel dt + D_\perp^{1/2}\frac{\nabla l\cdot\nabla\psi}{(rB)^2}dW_\perp \\ dV_\parallel = -\frac{\langle\mu\rangle_v}{m}\frac{\partial B}{\partial l}dt - \gamma v_\parallel dt + D_\parallel^{1/2}dW_\parallel \end{cases} \tag{5}$$

Here $D_\perp^{1/2}dW_\perp = -\partial_\vartheta\delta\varphi$, $mD_\parallel^{1/2}dW_\parallel = -q\partial_l\delta\varphi$, γ is the friction coefficient, and m the electron mass. Finally, $\langle\mu\rangle_v$ is the ensemble average over the velocity space of the magnetic moment.

6. The Fokker-Planck Equation

Using equations (4-5) and the formalism of stochastic analysis it is possible to derive the FPE of inward diffusion as below ($q = \nabla l\cdot\nabla\psi/(rB)^2$):

$$\begin{aligned} \frac{\partial P}{\partial t} &= -v_\parallel\frac{\partial P}{\partial l} + \frac{\partial}{\partial v_\parallel}\left[\left(\frac{\langle\mu\rangle_v}{m}\frac{\partial B}{\partial l} + \gamma v_\parallel\right)P\right] \\ &-\left(\frac{1}{2}-\alpha\right)D_\perp\frac{\partial}{\partial\psi}\left\{\left[\left(q\frac{\partial}{\partial l}+\frac{\partial}{\partial\psi}\right)lnrB\right]P\right\} \\ &+\frac{D_\parallel}{2}\frac{\partial^2 P}{\partial v_\parallel^2} + \frac{D_\perp}{2}\frac{\partial^2}{\partial l^2}(q^2P) + D_\perp\frac{\partial^2}{\partial l\partial\psi}(qP) \\ &+\frac{D_\perp}{2}\frac{\partial^2 P}{\partial\psi^2} - \alpha D_\perp\frac{\partial}{\partial l}\left(\frac{\partial q}{\partial\psi}P\right) - \alpha D_\perp\frac{\partial}{\partial l}\left(q\frac{\partial q}{\partial l}P\right) \end{aligned} \tag{6}$$

7. Numerical Simulations

Equation (6) was simulated for $\alpha = 0$. Figure 1 shows the relationship between the particle density ρ and the diffusion coefficient of inward diffusion $D_\perp$. In Fig. 2 the time evolution of the (equatorial) density maximum ρ_M and its radial position r_M is shown. As evident and in contrast with usual diffusion, inward diffusion steepens the density gradient along the equator of the dipole field.

8. Conclusion

A model for inward diffusion was developed through a stochastic approach. The phenomenon is interpreted as Fickian diffusion occurring in the proper frame, a reference system where particles perform Brownian motion with drift. Through numerical simulations of the derived FPE, the effect of normal transport in the formation of the self-organized plasma structure was studied. The first interesting result is that the particle number per unit fluxtube volume is not a constant, i.e. $\partial_\psi P \neq 0$. Moreover, because of the observed dependence of the peak position on normal transport, we expect inward diffusion to drive the plasma out of a tokamak confining device.

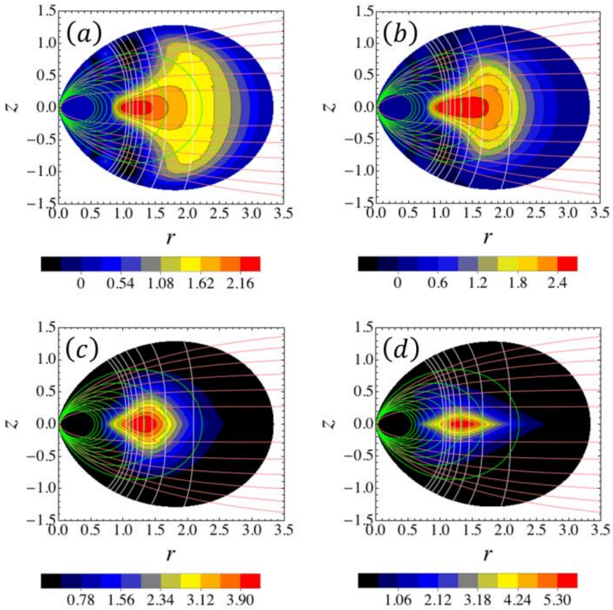

Fig. 1. Normal diffusion dependence of the laboratory density. (a): $10^{-2}D_\perp$ at $\mathrm{t} = 14.4$ and $\sim 84\%$ particles left. (b): $10^{-1}D_\perp$ at $\mathrm{t} = 7.2$ and $\sim 67\%$ particles left. (c): $D_\perp$ at $\mathrm{t} = 2.4$ and $\sim 55\%$ particles left. (d): $10D_\perp$ at $\mathrm{t} = 0.36$ and $\sim 46\%$ particles left.

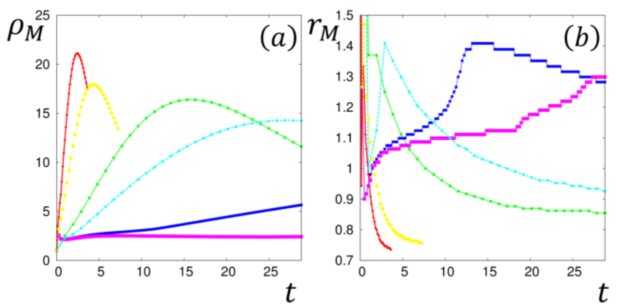

Fig. 2. (a): evolution of $\boldsymbol{\rho_M}$. (b): evolution of $\boldsymbol{r_M}$. Color legend: purple $10^{-2}D_\perp$, blue $10^{-1}D_\perp$, sky blue $0.5D_\perp$, green $D_\perp$, yellow $5D_\perp$, red $10D_\perp$.

9. References

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[1] The α dependent drift arising from Ito's Lemma in the l equation is neglected ($v_\parallel$ is assumed dominant).