Diffusion term implicitly appearing in the point vortex solution for two-dimensional inviscid Euler equation
2次元オイラー方程式における点渦解に現れる粘性

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Diffusion coefficient implicitly included in the point vortex solution for the two-dimensional inviscid Euler equation is examined analytically. This diffusive effect arises from a discrete distribution of the vorticity. The obtained diffusion coefficient includes a position correlation in addition to a time correlation. It can be regarded as an extension of the well-known Green-Kubo formula.

1. Introduction

To explain a large scale structure formation, for example, great red spot on Jupiter, eddy at Naruto, and typhoon, Onsager introduced a concept, "negative temperature" for the two-dimensional (2D) point vortex system [1]. If absolute temperature of a system is negative, there is more possible state at higher energy than lower energy as the probability is proportional to exp(-βH)) (H: system energy).

Much research effort has been devoted to understand the negative temperature state in the context of 2D turbulence [2-6]. One remarkable result may be an derivation of a mean field equation for the point vortex system. This equation is called sinh-Poisson equation [7]. Later, another member of Montgomery group reported that a time asymptotic distribution of 2D Navier-Stokes system at high Reynolds number reached a state predicted by the sinh-Poisson equation. This implies that an equilibrium state for the viscous Navier-Stokes system is similar to one for the inviscid point vortex system and that the point vortex system may have a diffusive effect.

On the other hand, in the review article of the point vortex method, Leonard said that "It now appears that using an increased number of point vortices of decreased strength will not yield a converged solution. ... Ironically, best results with the point vortex method often are achieved by using only a few vortices with a diffusive time integration scheme.[8]" This statement also implies a diffusive effect implicitly included in the point vortex system.

From these backgrounds, we have started an analytical estimate of the discussion coefficient for the 2D point vortex system.

The organization of this paper is as follows. First, we introduce the point vortex system as a microscopic solution. Second, the main result of the diffusion coefficient is presented. At last, we discuss the result.

2. Euler Equation and Klimontovich Equation

The 2D Euler equation (vorticity equation)

$$\frac{\partial \omega_z}{\partial t} + \boldsymbol{u} \cdot \nabla \omega_z = 0 \qquad (1)$$

has an discrete solution:

$$\omega_z(\boldsymbol{r}, t) = \sum_i \Omega_i \delta\left(\boldsymbol{r} - \boldsymbol{r}_i(t)\right) \qquad (2)$$

where $\boldsymbol{u}(\boldsymbol{r},t)$ and $\omega_z(\boldsymbol{r},t)$ are the velocity field and the vorticity field. The solution (2) is called the point vortex solution. A position and strength of the i-th point vortex is given by $\boldsymbol{r}_i = \boldsymbol{r}_i(x_i, y_i)$ and Ω_i. The value of Ω_i is either Ω_0 or -Ω_0 where Ω_0 is a positive constant. The vorticity field is discretized by the Dirac delta function $\delta(\boldsymbol{r} - \boldsymbol{r}_i)$.

Usually, fluid equation that describes macroscopic phenomena has a macroscopic smooth solution. However, Eq. (2) is not a smooth solution. Thus we consider the solution (2) is not a solution for the macroscopic Euler equation but a solution for a microscopic Euler equation whose form is identical with Eq. (1). To distinguish the microscopic equation from the macroscopic equation, we introduce a notation "hat". A variable with the hat means it microscopic.

$$\frac{\partial \hat{\omega}_z}{\partial t} + \hat{\boldsymbol{u}} \cdot \nabla \hat{\omega}_z = 0 \qquad (3)$$

Similar situation can be found in plasma physics. Time evolution of an exact phase space density

$$\hat{f}(\boldsymbol{r}, \boldsymbol{v}, t) = \sum_i \delta(\boldsymbol{r} - \boldsymbol{r}_i(t))\delta(\boldsymbol{v} - \boldsymbol{v}_i(t)) \qquad (4)$$

is given by the Klimontovich(-Dupree) equation

$$\frac{\partial \hat{f}}{\partial t} + v \cdot \nabla \hat{f} + \frac{q}{m}\left(\hat{\boldsymbol{E}} + \boldsymbol{v} \times \hat{\boldsymbol{B}}\right) \cdot \frac{\partial \hat{f}}{\partial \boldsymbol{v}} = 0 \quad . \qquad (5)$$

Taking an ensemble average of Eq. (5) yields a kinetic equation for the averaged phase space density, for example, Fokker-Planck equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = \frac{\partial}{\partial \boldsymbol{v}} \cdot \left(\vec{\vec{D}} \cdot \frac{\partial f}{\partial \boldsymbol{v}}\right) , \qquad (6)$$

and in the inviscid limit, Vlasov equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m}\left(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}\right) \cdot \frac{\partial f}{\partial \boldsymbol{v}} = 0 \quad . \qquad (7)$$

Note that the Vlasov equation is the inviscid equation in approximation.

We consider that the microscopic Euler equation that has the point vortex solution corresponds to the Klimontovich equation. The macroscopic, inviscid Euler equation corresponds to the Vlasov equation. There is no corresponding equation to the Fokker-Planck equation. Thus we derive it analytically using the Klimontovich formalism [9].

3. Diffusion Coefficient for the Point Vortex System

The starting equation is the microscopic Euler equation (3). We assume each microscopic variable consists of an averaged macroscopic part and a fluctuation.

$$\begin{aligned} \hat{\omega}_z(\boldsymbol{r}, t) &= \left\langle \hat{\omega}_z(\boldsymbol{r}, t) \right\rangle + \delta\hat{\omega}_z(\boldsymbol{r}, t) \\ &= \omega_z(\boldsymbol{r}, t) + \delta\hat{\omega}_z(\boldsymbol{r}, t) \end{aligned} \qquad (8)$$

Substituting Eq. (8) into Eq. (3) and averaging it, we obtain the following macroscopic Euler equation including a diffusion term in the right hand side:

$$\frac{\partial}{\partial t}\omega_z(\boldsymbol{r}, t) + \nabla \cdot \left[\boldsymbol{u}(\boldsymbol{r}, t)\omega_z(\boldsymbol{r}, t)\right] = -\nabla \cdot \left\langle \delta\hat{\boldsymbol{u}}(\boldsymbol{r}, t)\delta\hat{\omega}_z(\boldsymbol{r}, t) \right\rangle \qquad (9)$$

To evaluate Eq. (9), we introduce an linearized equation for the fluctuation.

$$\frac{\partial}{\partial t}\delta\hat{\omega}_z(\boldsymbol{r}, t) + \boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla\delta\hat{\omega}_z(\boldsymbol{r}, t) = -\delta\hat{\boldsymbol{u}}(\boldsymbol{r}, t) \cdot \nabla\omega_z(\boldsymbol{r}, t) \qquad (10)$$

This equation can be integrated:

$$\delta\hat{\omega}_z(\boldsymbol{r}, t) = -\int_{-\infty}^{t} d\tau \delta\hat{\boldsymbol{u}}(\boldsymbol{r} - (t - \tau)\boldsymbol{u}(\boldsymbol{r}, t), \tau) \qquad (11)$$

The resulting formula is given by

$$\begin{aligned} &\frac{\partial}{\partial t}\omega_z(\boldsymbol{r}, t) + \boldsymbol{u}(\boldsymbol{r}, t) \cdot \nabla\omega_z(\boldsymbol{r}, t) = -\nabla \cdot \left(\vec{\vec{\eta}} \cdot \nabla\omega_z\right) \\ &\vec{\vec{\eta}} = \int_{-\infty}^{t} d\tau \left\langle \delta\hat{\boldsymbol{u}}(\boldsymbol{r}, t)\delta\hat{\boldsymbol{u}}(\boldsymbol{r} - (t - \tau)\boldsymbol{u}, \tau) \right\rangle \end{aligned} \qquad (12)$$

4. Discussion

The resulting equation is an extension of the well-known Green-Kubo formula. In our result position correlation due to the macroscopic flow u is included in addition to time correlation. It may be possible to evaluate the diffusion term numerically.

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