Theory of action-angle variables for continuous spectrum in flowing plasma
流れをもつプラズマの連続スペクトルに対する作用・角変数理論

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Development of linear and nonlinear mode coupling theory is essential for the stability analysis of flowing plasmas, where the action-angle representation of linear fluctuation is a fundamental issue in the context of classical mechanics. Existence of continuous spectrum is however the greatest obstacle originating from the infinite dimensionality of plasmas. This work has made it possible to perform the action-angle representation of the contiuous spectrum with the use of appropriate mathematical techniques. This formalism also clarifies the wave energy for each eigenmode and continuum mode, where negative energy mode is generally responsible for onsets of various instabilities in flowing plasmas. Resonance between an eigenmode and a continuum mode can be viewed as a mode coupling and the signs of their wave energies identify whether the coupling leads to an instability or not. This unified viewpoint is applicable to various resonant phenomena including the Landau damping, the Alfvén resonance and the critical layor instability in shear flow.

1. Introduction

Analysis of waves and stabilities in flowing plasmas is important towards the understanding of efficient magnetic confinement in fusion devices and unresolved dynamics and transport in astrophysical plasmas. In the presence of equilibrium flow (e.g., rotation and shear flow), clear-cut stability criterion is hardly derived because the instability mechanism becomes highly involved. For instance, the Doppler shift of frequency causes both couplings and decouplings among various modes depending on the flow profile. Such the complicated stability problem cannot be treated by the energy principle [1] which has been successful for the stability analysis of static plasmas. According to classical mechanics with finite degree of freedom, an instability is triggered by resonant interaction between two eigenmodes carrying opposite signs of perturbation energies (Krein's theorem). On the other hand, since plasmas are infinite dimensional systems, the eigenfrequency of linear perturbation is often occupied by the continuous spectrum (which corresponds to a sum of infinite number of singular eigenmodes, or simply called the continuum mode). Although it is well known that resonance with the continuous spectrum strongly affects linear stability, even the definition of wave energy for continuous spectrum has not been so clear except for the quite limited cases [2], which seems to prevent further rigorous developments of linear and nonliner mode coupling theory in flowing plasmas.

In this work, we direct our attention to generalizing the notion of "action-angle variables" to the continuum mode, in order to derive the energy and momentum carried by it. Based on this Hamiltonian description, we have developed systematic methods for clarifying dynamical features (stabilities, effects of dissipation and nonlinear couplings etc.) of fluctuations in plasmas. Our achievements are summarized in three parts as follows.

2. Action-angle representation of continuum mode

We have established a general method for deriving the action-angle variables of continuum modes in dissipationless plasmas by means of the Laplace transform and the hyperfunction theory [3]. This method can decompose the energy of linear perturbation (or the variation $\tilde{H}$ of Hamiltonian) into the sum of wave energies for eigenmodes and continuum mode;

$$\tilde{H} = \sum_n \omega_n \mu_n + \int_\sigma \omega \mu(\omega) d\omega,$$

where $\omega_1, \omega_2, \ldots \in \mathbb{C}$ and an interval $\sigma \subset \mathbb{R}$ are respectively assumed to be discrete and continuous spectra (i.e., mode frequencies). It is the values $\mu_1, \mu_2, \ldots$ and the function $\mu(\omega)$ that signify the "action variables" for each mode (the angle variables simply correspond to the phase angles). While they are equivalent to the so-called "wave action" in the wave kinetic theory [4] when taking the eikonal (or short wave-length) limit, our theory is widely applicable to any eigenmode and continuum mode for which the wave number is not always well-defined along the background nonuniformity. As an application of our method, we have derived the action-angle variables of the Alfvén and sound continuous spectra in magnetohydrodynamics (MHD) and shown that there exists continuum mode with negative energy $[\omega\mu(\omega) < 0]$ in flowing plasma [5].

3. Stability of resonance between eigenmode and continuum mode

Once the energy of the continuum mode is clarified, we can discriminate linear stability of resonant interaction between eigenmode and continuum mode (in analogy with Krein's theory for finite dimensional systems). We have shown that the interaction leads to the resonant damping ($\simeq$ Landau damping) of the eigenmode if the continuum mode has the same sign of energy as the eigenmode, whereas it leads to the resonant growth ($\simeq$ inverse Landau damping) if the continuum mode has the opposite sign of energy [3, 6]. In addition, the negative energy mode tends to be destabilized by the effect of dissipation. We have adapted these theories to the stabilizing effect of flow on the resistive wall mode [7], where the stable regime appears when the resonant damping overcomes the dissipation-induced instability.

4. Adiabatic invariance of action variables

We have shown that the action variables of eigenmode and continuum mode are invariant when the mean fields are slowly varying compared with mode frequencies [6]. For example, the Landau damping can be understood as a mode conversion from an eigenmode to a continuum mode, where the wave action of the eigenmode μ_0 is absorbed into that of continuum mode $\int \mu(\omega) d\omega$. The damping rate is closely related to the spectral linewidth (equal to phase mixing rate) of the resultant continuum mode, which can be estimated by the invariance of wave-action without invoking the conventional analytic continuation of the dispersion relation [8]. This another viewpoint of the Landau damping is generically valid for the coupling between eigenmode and continuum mode in other dynamical systems (such the Alfvén resonance in MHD and the critial layor instability in shear flow).

Development of advanced theory of nonlinear mode coupling is also in progress based on this action-angle formalism. Note that the conventional wave-kinetic theory and the weak turbulence theory deal with three-wave resonance in wavenumber space (which is essentially resonance among plane waves). On the contrary, our method can precisely discuss the energy and momentum exchange among the eigenmodes and the continuum modes, which is clearly more appropriate for global (or long wave-length) fluctuation when the inhomogeneity of mean field is unignorable and plays a role of free energy source of fluctuation.

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