Quaternion Analysis of a Direct Matrix Converter Based on Space-Vector Modulation*)

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In a three-phase matrix converter based on space-vector modulation (SVM), nine switches are controlled so that the instantaneous space vector of the line-to-line voltage rotates smoothly in two-dimensional space. The quaternion is a four-dimensional hypercomplex number that is good at describing three-dimensional rotation, such as that seen in three-dimensional game graphics programming theory. Utilizing the quaternion capability, we analyze a matrix converter by three-dimensional rotation instead of transforming to two-dimensional rotation in alpha-beta coordinates. It was clarified that the projection of the quaternion locus in three-dimensional space in the (1,1,1) direction is the same as an alpha-beta transformation locus in two-dimensional space. Concerning the direct matrix converter, we clarified that the (1,1,1)-directional superposition of three-fold higher harmonics cannot be eliminated. The quaternion can rotate and divide a three-dimensional vector. When the output voltage quaternion is divided by input one, the switching quaternion is obtained. The quaternion characteristics will be utilized to analyze a matrix converter based on direct SVM in more detail.

Keywords: circulant matrix, quaternion, direct matrix converter, space vector modulation, three-phase to three-phase

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1. Introduction

A matrix converter can produce an output voltage waveform of arbitrary frequency and phase angle, and can be utilized as a power supply for a resonant magnetic perturbation (RMP) coil. The converter can produce an input current waveform of an arbitrary phase angle and enable unity input power factor, which makes a fusion reactor more efficient.

In the three-phase indirect matrix converter, a voltage source inverter and voltage source regulator were combined and their individual technology was utilized [1]. In the three-phase direct matrix converter based on space-vector modulation (SVM), nine switches are controlled so that an instantaneous space vector of the line-to-line voltage rotates smoothly in two-dimensional space [2].

The quaternion, a four-dimensional hypercomplex number, is good at describing three-dimensional rotation and has been utilized in three-dimensional game graphics programming theory. Utilizing the quaternion capability, we analyze a matrix converter by three-dimensional rotation instead of transforming to two-dimensional rotation in alpha-beta coordinates.

It was clarified that the projection of the quaternion locus in three-dimensional space in the (1,1,1) direction is the same as the alpha-beta transformation locus in two-dimensional space. Concerning the direct matrix converter, we clarified the (1,1,1)-directional superposition of three-fold higher harmonics, which is necessary for improvement of the voltage transformation ratio.

The quaternion can rotate and divide a three-dimensional vector. When an output voltage quaternion is divided by input one, the switching quaternion is obtained. Quaternion characteristics will be utilized to analyze a matrix converter based on direct SVM in more detail.

2. Space Vector and Quaternion

Let us consider the transformation from a star-connection three-phase electromotive force (phase voltage) to a line-to-line voltage:

$$\begin{bmatrix} e_{ab} \\ e_{bc} \\ e_{ca} \end{bmatrix} = \begin{bmatrix} +1 & -1 & 0 \\ 0 & +1 & -1 \\ -1 & 0 & +1 \end{bmatrix} \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix}. \tag{1}$$

Since this transformation matrix is a cyclic matrix, the eigenvalues are $1-1$, $1-\exp(j2\pi/3)$, and $1-\exp(j4\pi/3)$.

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Therefore, we can consider a complex transformation wherein the matrix is composed of the eigenvectors. We can define $\alpha\beta 0$ transformation by considering the real part:

$$\begin{bmatrix} e_\alpha \\ e_\beta \\ e_0 \end{bmatrix} = \sqrt{\frac{2}{3}} \begin{bmatrix} \cos(0) & \cos(2\pi/3) & \cos(4\pi/3) \\ \sin(0) & \sin(2\pi/3) & \sin(4\pi/3) \\ 1/\sqrt{2} & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} e_a \\ e_b \\ e_c \end{bmatrix}. \tag{2}$$

Therefore, we can consider that the three-phase voltage vector is transformed to the orthogonal coordinates shown in Fig. 1. We can define the space vector by considering the projection to the x-y plane (two-dimensional complex plane).

To represent three-phase AC in three dimensions, we introduce the quaternion (a four-dimensional hypercomplex number), which is extended from a two-dimensional complex number [3]:

$$q = a + v = a + (iv_x + jv_y + kv_z), \tag{3}$$
$$i^2 = j^2 = k^2 = -1, \tag{4}$$
$$ij = -ji = k, \tag{5}$$
$$jk = -kj = i, \tag{6}$$
$$ki = -ik = j. \tag{7}$$

The quaternion is divided into real part (scalar part) a and imaginary part (vector part) v, similar to a two-dimensional complex number. Namely, the vector part has properties of a vector, where imaginary numbers i, j, and k behave as if they are unit base vectors, but they also have properties of a hypercomplex number. The square of imaginary numbers i, j, and k are equal to -1. The product of different imaginary numbers is the other imaginary number, the sign depends on the order, and commutative law does not hold. To assign three-phase AC to the vector part, we consider the exponential representation of the quaternion:

$$q = a + \hat{n}\|v\| = \|q\|(\cos\theta + \hat{n}\sin\theta) = \|q\|\epsilon^{\hat{n}\theta}, \tag{8}$$
$$\hat{n} = (iv_x + jv_y + kv_z)/\|v\|, \tag{9}$$
$$\|v\|^2 = (v_x)^2 + (v_y)^2 + (v_z)^2, \tag{10}$$
$$\|q\|^2 = a^2 + \|v\|^2. \tag{11}$$

The quaternion can manipulate four dimensions, as it is interpreted as a four-dimensional number. But if we let the scalar part be equal to zero, we can consider the left-hand-side product of the exponential hypercomplex number ($\|q\| = 1, \hat{n} = k$) and a vector on the x-y plane. Namely, when the exponential number is multiplied to the vector part from the left-hand side, the vector part rotates by θ in the counter-clockwise direction with an axis of the unit vector $\hat{n}$. Here, the rotating axis must be perpendicular to the vector. When the vector has a component parallel to $\hat{n}$, the vector rotates on the $(1, \hat{n})$ plane and the scalar part appears.

We can assign three-phase AC phase (line-to-neutral) voltages to the vector part of the quaternion:

$$e = \sqrt{2}E \begin{Bmatrix} +i\cos(\omega t - 0\pi/3) \\ +j\cos(\omega t - 2\pi/3) \\ +k\cos(\omega t - 4\pi/3) \end{Bmatrix} = \epsilon^{\hat{n}\omega t}\sqrt{3}Ee_0, \tag{12}$$
$$\hat{n} = (+i + j + k)/\sqrt{3}, \tag{13}$$
$$e_0 = (+i1 - j1/2 - k1/2)/\sqrt{3/2}. \tag{14}$$

Namely, the vector part of the quaternion represents the initial three-phase (positive sequence) AC voltage vector $\sqrt{3}Ee_0$, which rotates in the counter-clockwise direction with an axis of unit vector $\hat{n}$. In this case, the locus of the rotating vector is a circle on the plane, which is perpendicular to $\hat{n}$ and includes the origin. While we look at the x-y plane from the z-axis in the case of a space vector (Fig. 1), we look at the (1,1,1) plane from the (1,1,1) direction in the case of the quaternion (after-mentioned Fig. 5).

3. Realization of the Switching Matrix

A three-phase to three-phase matrix converter (Fig. 2) is discussed, when the switching frequency is much higher than the modulation frequency.

Usually, in a vector equation with coefficient matrix A, we can solve the equation $Ax = y$ by obtaining the inverse matrix A^{-1}. But in complex number equations, we can calculate the transformation $C = v/u$ by dividing the number v by the number u.

Similarly, in the switching equation for a matrix con-

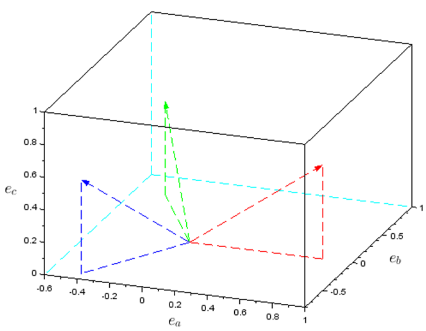

Fig. 1 Orthogonal coordinate system of $\alpha\beta 0$ element method.

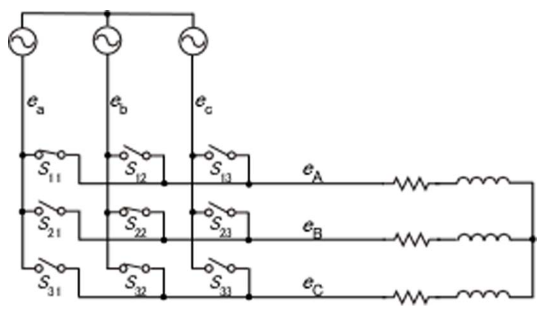

Fig. 2 Direct-type matrix converter. The input voltage source is shown at the top and the output load is shown at the right, so that the switching configuration corresponds to the switching matrix element.

verter, we can express the three-phase input and output voltages using the quaternion:

$$\exp(+\hat{n}\omega_o t)\,\sqrt{3}E_o e_0 = Q\exp(+\hat{n}\omega_i t)\,\sqrt{3}E_i e_0. \quad (15)$$

In this case, the switching quaternion Q is calculated as follows:

$$\begin{aligned} Q &= \exp(+\hat{n}\omega_o t)(E_o/E_i)\exp(-\hat{n}\omega_i t) \\ &= r\exp\{+\hat{n}(\omega_o-\omega_i)t\}. \end{aligned} \quad (16)$$

Here, r is the voltage transfer ratio. The above quaternion equation is re-expressed by the vector equation with a switching matrix [4, 5]:

$$\begin{aligned} &\exp(+\hat{n}\omega_o t)\,\sqrt{3}E_o e_0 \\ &\quad = r\exp\{+\hat{n}(\omega_o-\omega_i)t\}\exp(+\hat{n}\omega_i t)\,\sqrt{3}E_i e_0, \quad (17) \\ &E_o\begin{bmatrix}\cos(\omega_o t-0\pi/3)\\ \cos(\omega_o t-2\pi/3)\\ \cos(\omega_o t-4\pi/3)\end{bmatrix} \\ &\quad = r\begin{bmatrix}\cos(\omega_m t) & -\frac{1}{\sqrt{3}}\sin(\omega_m t) & +\frac{1}{\sqrt{3}}\sin(\omega_m t)\\ +\frac{1}{\sqrt{3}}\sin(\omega_m t) & \cos(\omega_m t) & -\frac{1}{\sqrt{3}}\sin(\omega_m t)\\ -\frac{1}{\sqrt{3}}\sin(\omega_m t) & +\frac{1}{\sqrt{3}}\sin(\omega_m t) & \cos(\omega_m t)\end{bmatrix} \\ &E_i\begin{bmatrix}\cos(\omega_i t-0\pi/3)\\ \cos(\omega_i t-2\pi/3)\\ \cos(\omega_i t-4\pi/3)\end{bmatrix}, \quad (18) \\ &\omega_m = \omega_o-\omega_i. \quad (19) \end{aligned}$$

For all components to be larger than 0 and smaller than 1, we have only to multiply by 1/2 and add 1/2. However, the summation of three elements in each row is not constant and may not be made unity by any means as far as we consider the phase voltage.

By considering the line-to-line voltage, we can deduce the original Venturini method [6]:

$$\begin{aligned} &V_o\begin{bmatrix}\cos(\omega_o t-0\pi/3)\\ \cos(\omega_o t-2\pi/3)\\ \cos(\omega_o t-4\pi/3)\end{bmatrix} = \begin{bmatrix}0 & +1 & -1\\ -1 & 0 & +1\\ +1 & -1 & 0\end{bmatrix} \\ &r\begin{bmatrix}\cos(\omega_m t) & -\frac{1}{\sqrt{3}}\sin(\omega_m t) & +\frac{1}{\sqrt{3}}\sin(\omega_m t)\\ +\frac{1}{\sqrt{3}}\sin(\omega_m t) & \cos(\omega_m t) & -\frac{1}{\sqrt{3}}\sin(\omega_m t)\\ -\frac{1}{\sqrt{3}}\sin(\omega_m t) & +\frac{1}{\sqrt{3}}\sin(\omega_m t) & \cos(\omega_m t)\end{bmatrix} \\ &\begin{bmatrix}0 & -1 & +1\\ +1 & 0 & -1\\ -1 & +1 & 0\end{bmatrix} V_i\begin{bmatrix}\cos(\omega_i t-0\pi/3)\\ \cos(\omega_i t-2\pi/3)\\ \cos(\omega_i t-4\pi/3)\end{bmatrix}, \\ &\Rightarrow \frac{1}{3}\begin{bmatrix}1{+}2r\sin(\omega_m t+0\pi/3) & 1{+}2r\sin(\omega_m t+2\pi/3) & \ldots\ldots\\ 1{+}2r\sin(\omega_m t+4\pi/3) & 1{+}2r\sin(\omega_m t+0\pi/3) & \ldots\ldots\\ 1{+}2r\sin(\omega_m t+2\pi/3) & 1{+}2r\sin(\omega_m t+4\pi/3) & \ldots\ldots\end{bmatrix} \\ &V_i\begin{bmatrix}\cos(\omega_i t-0\pi/3)\\ \cos(\omega_i t-2\pi/3)\\ \cos(\omega_i t-4\pi/3)\end{bmatrix}. \quad (20) \end{aligned}$$

Concerning the improved Venturini method, third higher harmonics of the desired output phase voltage and input phase voltage can be added, since the harmonics constitute zero space about the transformation from phase voltage to line-to-line voltage.

4. Analysis of Direct Matrix Converter

The three-phase to three-phase direct matrix converter based on SVM is discussed in Ref. [2]. There are 27 kinds of switching configurations, since any output phase must not be opened in case of an inductive load, such as a motor. Input phases a, b, and c are expressed as phases 1, 2, and 3, respectively, for considering the switching configuration mathematically. A switching matrix is defined as a matrix, which is composed of S_{ij}. To study its characteristics, we define a configuration (column) vector, which is composed of column number j such as $S_{ij} = 1$, and define a configuration matrix, composed of the configuration (column) vectors. In the case of Fig. 2, the switching matrix is such a matrix that $S_{ij} = 0$, except for $S_{11} = 1, S_{22} = 1$, and $S_{32} = 1$: the configuration vector is $(1, 2, 2)^T$ where the superscript T denotes the transpose. By connecting the closed switches, we can express the switching matrix with a bent line (+1) in Fig. 3. Similarly, switching matrices (+2) and (+3) of the first group are expressed in Fig. 3 and the configuration vectors are $(2, 3, 3)^T$ and $(3, 1, 1)^T$, respectively. The switching matrix of the first group of the three switching configurations is singular-value-decomposed as follows:

$$\begin{aligned} &\begin{bmatrix}1 & 2 & 3\\ 2 & 3 & 1\\ 2 & 3 & 1\end{bmatrix} = \begin{bmatrix}-0.54 & +0.84 & 0\\ -0.59 & -0.38 & -1/\sqrt{2}\\ -0,59 & -0.38 & +1/\sqrt{2}\end{bmatrix} \\ &\begin{bmatrix}6.2 & 0 & 0\\ 0 & 2.0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-0.37 & -0.65 & -0.67\\ +0.46 & +0.50 & -0.74\\ -0.81 & -0.58 & +0.12\end{bmatrix}. \end{aligned} \quad (21)$$

Since the left-singular vector corresponding to the singular value 0 is perpendicular to the base vector (phase A), the switching produces the base vector (phase A). Among the three switching configurations, the input current of phase 1 (phase a) is controlled by 122 and 311 (named as +1 and +3) switching configurations. Similarly, the second base vector is produced by 221 and 113 (named as +7 and +9) switching configurations. The output voltage space vector is composed of the four switching configurations as their

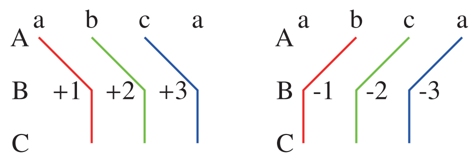

Fig. 3 Switching configurations in the direct matrix converter. The first positive group is composed of (+1), (+2), and (+3). The first negative group is composed of (−1), (−2), and (−3).

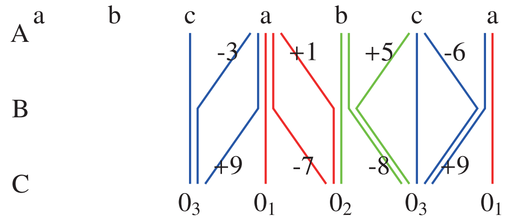

Fig. 4 Symmetric switching sequence with zero switching $(0_3, -3, +9, 0_1, -7, +1, 0_2)$ in the first sector and $(0_2, -8, +5, 0_3, -6, +9, 0_1)$ in the second sector in the direct matrix converter.

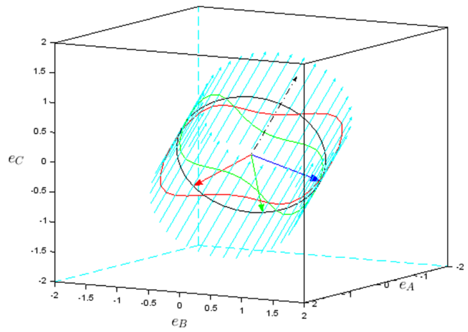

Fig. 5 Output phase voltage quaternion in the direct matrix converter. The output line voltage quaternion (a circle) is also shown.

linear combination. The weight of the linear combination (duty ratio) is determined from the four-series equations.

The duty ratio is determined by transferring the four switching configurations as shown in Fig. 4. The red line indicates a base vector of phase a, the green line phase b, and the blue line phase c. In the case of an asymmetric switching sequence, $(-3, +9, 0_1, -7, +1)$ and $(+1, -7, 0_1, +9, -3)$ are repeated in the first sector. During this switching in the first sector, the output phase A is always connected to input phase a and the other switch commutates only one time between adjacent configurations. Similarly, $(-8, +5, 0_3, -6, +9)$ and $(+9, -6, 0_3, +5, -8)$ are repeated in the second sector. During this switching in the second sector, the output phase C is always connected to input phase c. In the case of an asymmetric switching sequence, the switch must commutate twice between adjacent sectors. In the case of a symmetric switching sequence, by adding zero switching (0_2), the switch only has to commutate only once even between adjacent sectors, as shown in Fig. 4.

The output phase voltage quaternion is shown in Fig. 5 in the case of symmetric switching with three 0 configurations $0_1, 0_2, 0_3$. The quaternion locus in the first sector (left half of Fig. 4) is drawn by a red line and the second sector (right half) by a green line. The quaternion locus is continuous at the phase angle $\pi/3$ (the intersection point) between the sectors. When only one 0 configuration 0_1 is adopted, the output phase voltage quaternion is shown in

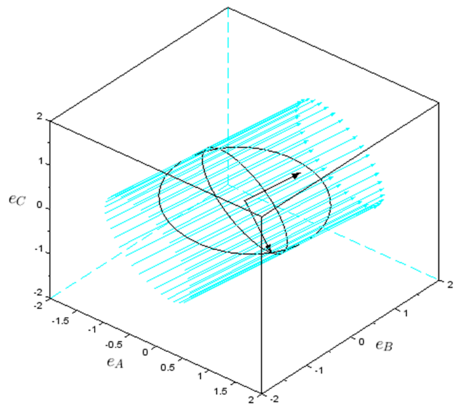

Fig. 6 Output phase voltage quaternion in the direct matrix converter. The only one 0 switching configuration 0_1 is adopted in asymmetric space vector modulation.

Fig. 6.

From Figs. 5 and 6, the output phase voltage quaternion seems to be located on the surface of a cylinder, which is extracted from the input phase voltage quaternion locus (a circle) in the (1,1,1) direction. The output line voltage quaternion draws a circle and the locus is expressed by the following equations:

$$v_{AB} + v_{BC} + v_{CA} = 0, \tag{22}$$

$$(v_{AB})^2 + (v_{BC})^2 + (v_{CA})^2 = (V_o)^2. \tag{23}$$

From the transforming matrix, the locus of the output phase voltage quaternion is expressed as follows:

$$(e_A - e_B)^2 + (e_B - e_C)^2 + (e_C - e_A)^2 = (\sqrt{3}E_o)^2. \tag{24}$$

Namely, the output phase voltage quaternion is located on the surface of a cylinder, which is extracted from the input phase voltage quaternion locus in the (1,1,1) direction.

5. Summary

Since a transformation matrix from three-phase AC phase voltage to line voltage is a circulant matrix, the three-phase AC phase voltage is decomposed into zero-sequence, positive-sequence and negative-sequence voltages. It was clarified that the projection of the quaternion locus in three-dimensional space in the $(1, 1, 1)$ direction is the same as an $\alpha\beta$ transformation locus in two-dimensional space.

The quaternion can rotate and divide a three-dimensional vector. When the output voltage quaternion is divided by input one, the switching quaternion is obtained. Though we could obtain a switching matrix, whose element is larger than 0 and smaller than 1, we could not obtain a switching matrix wherein the sum of its row elements is unity. By considering the line-to-line voltage, we can deduce the original Venturini method.

In the direct matrix converter, the output phase voltage quaternion does not draw a circle, and the locus is located on a cylindrical surface. This surface is obtained by extracting the input phase voltage quaternion locus (a circle) in the (1,1,1) direction.

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