

Sparsity-Promoting Dynamic Mode Decomposition of Plasma Turbulence

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A data-driven approach called sparsity-promoting dynamic mode decomposition (SP-DMD) is applied to the plasma turbulence signals obtained with an azimuthal probe array. The spatiotemporal turbulence can be reasonably decomposed into seven modes which capture the azimuthal bunching of the turbulence. A superiority of the DMD analysis to the conventional stationary analysis is demonstrated.

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Time-varying structures are frequently observed in non-equilibrium systems. Magnetically confined plasma is a typical non-equilibrium system. The spatial inhomogeneity of plasma, which is produced and sustained by external sources, excites instabilities in the plasma. In turn, transport driven by the instabilities tries to mitigate the plasma's inhomogeneity. As a result, spatial gradients, waves, flows and eddies coexist in the plasma and interact with each other causing the plasma to become turbulent [1, 2]. Due to the various nonlinearities in plasma, multi-scale spatiotemporal structures can form in the plasma turbulence [3]. The identification of the spatiotemporal structures in plasma turbulence is essential for visualizing their coarse-grained nature, where many decomposition techniques have been applied to time-series data of plasma turbulence [4, 5]. Fourier analysis is frequently used as a decomposition technique, however, ensemble averaging is required when it is applied to random signals. Usually, ensemble averaging is replaced by time averaging [6], which can mask the non-stationarity of the signals. Actually, structure in plasma turbulence varies in time [7], and thus new analysis tools, which can extract the non-stationary spatiotemporal structure of the turbulence, are required. Here we apply a dynamic mode decomposition (DMD) analysis to signals of turbulence in a magnetized laboratory plasma. Laboratory plasma is very useful as it allows us to verify the capability of our method, and because it has excellent reproducibility and controllability and allows for multi-point simultaneous measurements to be made.

Since it was introduced in 2008 [8], DMD has been applied to fluid flow studies especially in analyses of nonlinear dynamics [9, 10]. From the spatiotemporal data obtained from an experiment or a numerical simulation,

DMD extracts a spatial pattern with a temporal frequency and a growth/decay rate. This spatial pattern is called a DMD mode. In standard DMD, high-dimensional data is decomposed into a large number of DMD modes. This makes it difficult to interpret the extracted modes. To address this difficulty, we introduce a variant of DMD, called sparsity-promoting dynamic mode decomposition (SP-DMD) [11], which allows us to reconstruct the original data using a small number of dominant DMD modes. In the following, we describe how standard DMD and SP-DMD work.

In standard DMD, the following two matrices are given as input:

$$\Psi_0 := [\psi_0 \ \psi_1 \ \dots \ \psi_{N-1}] \in \mathbb{C}^{M \times N}, \text{ and} \quad (1)$$

$$\Psi_1 := [\psi_1 \ \psi_2 \ \dots \ \psi_N] \in \mathbb{C}^{M \times N}, \quad (2)$$

where $\psi_t \in \mathbb{C}^M$ is the observed value at time t , M is the number of observation points, and N is the number of time steps. If a linear time-invariant system is assumed, it holds that:

$$\psi_{t+1} \approx A\psi_t \quad (t = 0, \dots, N-1). \quad (3)$$

The matrix A best-fit to the observed data, which minimizes the Frobenius norm of $\Psi_1 - A\Psi_0$, is calculated as follows:

$$A := \Psi_1 \Psi_0^+, \quad (4)$$

where Ψ_0^+ is the pseudoinverse of Ψ_0 . From the economy-size singular value decomposition (SVD) $\Psi_0 = U\Sigma V^*$, it follows that:

$$\Psi_0^+ = V\Sigma^{-1}U^*. \quad (5)$$

If dimensionality reduction via the matrix U is adopted, the

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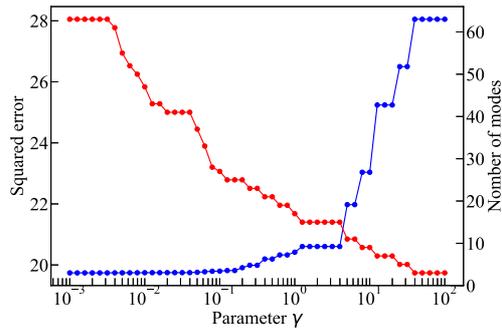


Fig. 1 Trade-off between the number of modes and the squared error.

state at time t is calculated as follows:

$$\psi_t \approx A^t \psi_0 \approx \sum_{i=1}^r \phi_i \mu_i^t \alpha_i, \quad (6)$$

where r is the rank of Ψ_0 , ϕ_i ($:= U y_i$) is the DMD mode, which is calculated from the eigenvector y_i of F ($:= U^* \Psi_1 V \Sigma^{-1}$), μ_i is a DMD eigenvalue, which is identical to the eigenvalue of F , and α_i is the amplitude, which depends on the initial state ψ_0 . To obtain a better amplitude vector $\alpha = (\alpha_1, \dots, \alpha_r)^T$, the objective function $J(\alpha)$ for minimizing the squared error between the observed data (Eq. (1)) and the calculation (Eq. (6)) can be defined.

In SP-DMD, by adding an l_1 -regularization term to the objective function, a new function, $J(\alpha) + \gamma \sum_{i=1}^r |\alpha_i|$, is minimized. Consequently, a small number of dominant DMD modes are selected. That is, only those modes that have non-zero amplitudes, while all the other modes have zero. After the selection, the amplitudes are computed again by minimizing $J(\alpha)$ under the condition that the amplitudes of the non-selected modes are fixed to zero.

In this study, the DMD analysis is applied to the signals of plasma turbulence obtained in a linear magnetized plasma device, LMD-U [12]. In a conventional study [12], the streamer structure [3], azimuthal self-bunching of drift waves (6.6 kHz and 7.8 kHz components), can be identified with a Fourier analysis. The streamer forms through the three-wave coupling (nonlinear phase locking) between the mediator (1.2 kHz components) and the carrier (drift waves). The turbulence structures that form in the LMD-U are quasi-coherent. Thus, stationary signal processing is able to capture the rough features of the spatiotemporal structures in the LMD-U plasma. Thus, this data-set is very useful as it allows us to verify the capability of our DMD analysis. In this analysis, the SP-DMD algorithm is applied to the spatiotemporal data under the condition that $\|\phi_i\| = 1$ ($i = 1, \dots, 64$). Here the time window is 1 ms.

Figure 1 shows the number of modes (N_{mode}) and the squared error (SE), *i.e.*, minimized $J(\alpha)$, versus the regularization parameter γ . When $\gamma = 0$, the result corresponds to that from the standard DMD. An increase in γ leads to a decrease in N_{mode} and an increase in the SE. In general,

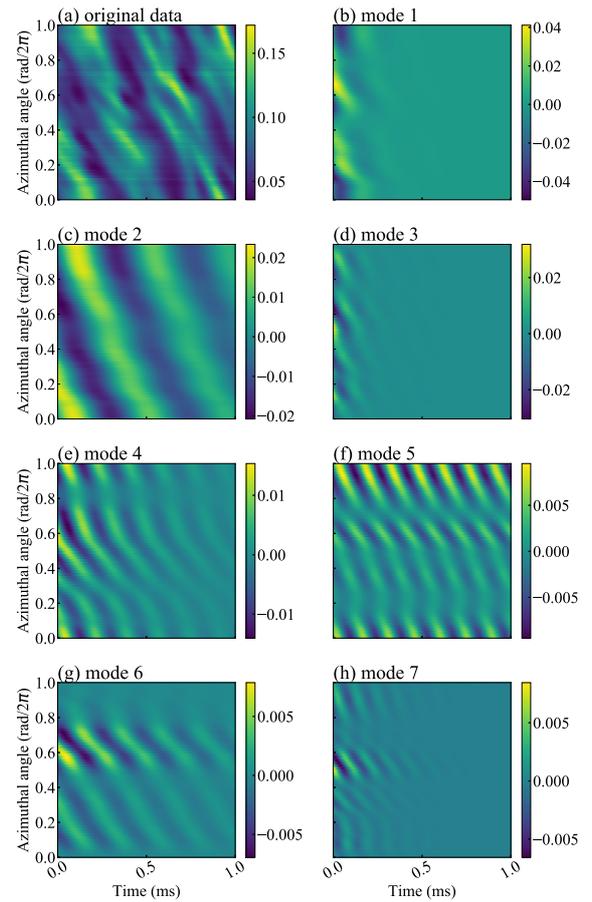


Fig. 2 Spatiotemporal patterns of the ion saturation currents measured with the 64-ch probe array. (a) Original data, (b) - (h) results of SP-DMD using a regularization parameter of $\gamma = 2$.

the presence of too many modes causes difficulty when interpreting the results. When $\gamma < 1$, N_{mode} decreases by about one third although the increase in the SE is moderate. Both N_{mode} and SE are stable for $1 < \gamma < 4$, while SE increases rapidly for $\gamma > 4$. Therefore, we consider that the stable interval ($1 < \gamma < 4$) is a reasonable selection when considering the trade-off between N_{mode} and the SE. In this interval, the number of modes is 15, where the corresponding DMD eigenvalues consist of one real number and seven pairs of conjugate complex numbers. Thus, the original data is characterized by one non-oscillatory mode and seven kinds of oscillatory modes.

Figure 2 (a) shows a heatmap of the ion saturation current measured with a 64-channel azimuthal probe array, and Figs. 2 (b) - (h) show the spatiotemporal patterns of the seven kinds modes for the parameter γ selected above. A drift wave, which is an azimuthally propagating wave, can be identified as the oblique stripe pattern seen in Fig. 2 (c), *i.e.*, as mode-2. Figure 2 (c) indicates that the amplitude of the drift wave gradually changes within a time-scale of 1 ms, which is a characteristic feature of turbulence.

As shown in Figs. 2 (e), (f) and (g), the amplitudes

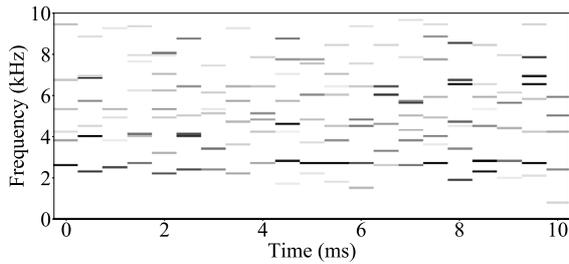


Fig. 3 High-frequency-resolution spectrogram.

of mode-4, mode-5 and mode-6 vary in the azimuthal direction. For example, the amplitude of mode-6 becomes strong around $\theta \sim 0.6$, where θ is the azimuthal angle normalized by 2π . In a different time window, amplitude enhancements appear at different values of θ , which seem to propagate in the azimuthal direction. Thus, these structures may be related to the azimuthal bunching of the drift waves [3, 12]. (This point will be discussed in the full paper.) In addition, the decay time-scale of mode-5 is longer than that of the drift wave. This is a characteristic of the mediator which plays an important role in energy transfer between drift waves [12].

Figure 2 also demonstrates the presence of different structures. Some of these structures (mode-1, mode-3 and mode-7) decay within 1 ms. These structures may be linearly stable but be excited transiently due to nonlinear energy transfer through the turbulence. Identification of energy transfer from/to these structures is left for future work but we emphasize that the SP-DMD analysis enables us to extract such fine spatiotemporal structures.

Also, a limitation of the SP-DMD should be discussed from a technical viewpoint. As we have discussed above, the SP-DMD method extracts decay modes successfully. However, the SP-DMD of plasma turbulence tends to miss the extraction of some growth modes from our observation, although plasma turbulence shows fluctuations which disappear and appear repeatedly as its nature. Therefore, the sum of all modes, *i.e.*, the reconstruction, diverges from the original data over time. This is the reason why we use a 1 ms window rather than wider one. To analyze a longer time period, we use the following sliding window method.

Next, the fluctuation of the frequency of each DMD mode was examined by an overlapping sliding window model with a size of 1 ms and a slide of 0.5 ms, *i.e.*, an overlap of 0.5 ms. Figure 3 shows the result for a 10 ms period. The intensity is denoted by the absolute value of the DMD amplitude, $|\alpha_i|$. The parameter γ is fixed to 2 according to the previous discussion. Again, please note that a mode within a window has only one temporal frequency corresponding to μ_i in Eq. (6), although the spatial pattern ϕ_i also has a phase difference resulting in the complicated spatiotemporal pattern observed in Fig. 2. In Fig. 3, it can be seen that most DMD modes have a frequency between 2 kHz and 10 kHz. Most of the lower

bound of the frequency appears in the spectrogram around 2 to 3 kHz with high intensity. For short-time Fourier transform (STFT), the frequency resolution is bound down to 1 kHz due to the uncertainty principle when the sliding window size is set to 1 ms, *i.e.*, the same size as in Fig. 3. In contrast, the frequency resolution obtained by the SP-DMD in Fig. 3 is much higher. This is because the DMD treats instantaneous frequency as the time derivative of each instantaneous phase. The DMD spectrogram is capable of capturing smaller fluctuations compared with the STFT. In addition to the time-varying frequency, the DMD can give us a time-varying amplitude. To present the preceding/following time relation or co-occurrence information of events is essential to identify the direction of energy transfer between the fluctuating structures. Again, we emphasize that a DMD analysis can extract fine spatiotemporal structures that are masked in the conventional Fourier analysis, which is considered to be important for more a comprehensive understanding of plasma turbulence dynamics.

To observe the non-stationary spatiotemporal structure in turbulence, a DMD analysis was applied to data obtained from a laboratory plasma experiment in which a streamer had been identified. The DMD analysis detected: i) a drift wave structure, ii) azimuthally bunched structures and iii) structures with very short time-scales (< 1 ms). Observations of the time-varying spatiotemporal structures in turbulence are important because turbulence is dynamic in its nature. The variant of DMD method used here was found to be very powerful and thus will be applied to other spatiotemporal structure decomposition problems in future works.

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