# Formulation of Two-Dimensional Transport Modeling in Tokamak Plasmas

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(Received 28 July 2013 / Accepted 2 December 2013)

A two-dimensional transport modeling applicable to a whole tokamak plasma is proposed. The model is derived from the multi-fluid equations and Maxwell's equations and the moment approach of neoclassical transport is employed as fluid closures. The multi-fluid equations consist of the equations for particle density, momentum, energy and total heat flux transport for each plasma species. The expressions of the parallel viscosity and heat viscosity are extended in order to be applicable to both inside and outside of the last closed flux surface. It is confirmed that our neoclassical transport model is consistent with the ordinary flux-surface-averaged one-dimensional neoclassical transport model. Our transport equations are coupled with the electromagnetic equations in order to describe the time evolution of tokamak plasmas. The procedure for coupling a transport solver based on our transport model with an equilibrium solver is also briefly described.



Keywords: two-dimensional transport modeling, neoclassical transport, multi-fluid equation

DOI: 10.1585/pfr.9.1403002

## 1. Introduction

The core and peripheral plasmas are strongly coupled with each other in tokamaks. The particle and heat fluxes from the core determine the behavior of the peripheral plasma, while the peripheral plasma determines the edge density and temperature, boundary conditions of the core plasma. The transport in the core and the peripheral regions, however, have been analyzed separately until recently owing to the difference in modeling configurations.

In most conventional transport analyses in the core region, transport is usually described as one-dimensional problem in the radial direction based on the magnetic flux surface average, since the transport along the field lines is so fast that the poloidal and toroidal dependences of the plasma quantities such as the particle density and the temperature are small. One-dimensional (1D) core transport codes or 1.5D core transport codes composed of an one-dimensional core transport module and a two-dimensional MHD equilibrium module have been used for analyzing various transport issues [1–8], comparison of turbulent transport models [5], analysis of edge transport mechanism [6], analysis of plasma rotation [8] and so on.

A standard core transport modeling consists of diffusion equations for particle, toroidal momentum and energy transport as well as poloidal magnetic field [9, 10]. The force and the energy-weighted force balances in the parallel direction are employed to determine the poloidal particle and heat fluxes in the neoclassical theory [11, 12] and the charge neutrality is assumed. A new core transport modeling [7, 8] has been introduced for the analysis of plasma rotation. It includes the equation of motion and the radial electric field in addition to those mentioned above and the charge neutrality is not assumed, since the rotation and the radial electric field are strongly coupled.

On the other hand, in the peripheral region, the transport is usually described as a two-dimensional problem on the poloidal cross-section, since variation of physical quantities along a field line is relatively large and important to understand the transport mechanism in the peripheral region. Two-dimensional (2D) peripheral transport codes, for example B2 [13], B2.5 [14], EDGE2D [15], UEDGE [16] and SONIC [17, 18] have been developed and integrated with the neutral particle transport code and atomic process data. They are used for various peripheral transport issues, impurity transport analysis [17], divertor designing [18], and so on. Since these analyses are mainly based on the collisional transport model, they are not directly applicable to the weakly collisional core plasmas.

A standard peripheral transport modeling consists of advection-diffusion equations for particle, parallel momentum and energy transport [9]. These are based on the Braginskii's equations [19] extended to multi-species plasma [13]. A new modeling [20] has been introduced for smooth extension to the weakly collision regime. It includes the contribution of the heat flux to the parallel viscosity term, since the contribution of the heat flux is comparable to that of the particle flux and important in the weakly collisional regime.

Recently, integrated core-peripheral transport simulations on the whole tokamak plasma have been done by coupling a 1.5D core transport code with a 2D peripheral transport code. The simulation with TOPICS-IB [6] and

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SONIC [17] analyzed a L-H transition in JT-60SA [21] and that with a integrated suite JINTRAC [22] also analyzed a consecutive ELM-crash in JET [23]. There is an ambiguity, however, in the connection at the computational boundary which is an appropriately chosen flux surface inside and near the last closed flux surface (LCFS).

In order to resolve this issue, the overlap computational domain in the edge region has been proposed [21]. Since simulation results may depend on the choice of the location of the boundary and the connection rule, a transport code applicable to a whole plasma is desired for consistent transport simulation in both core and peripheral plasmas. Some efforts have been devoted to two-dimensional transport modeling, though they have not been published yet.

In this paper, we formulate two-dimensional fluid transport equations including the neoclassical transport [11] in the magnetic surface coordinate system. Our model is applicable to both core and peripheral plasmas in the axisymmetric tokamak configuration.

This paper is organized as follows. In Sec. 2, the property and advantage of the magnetic surface coordinate system are described. The orderings used in this paper is discussed in Sec. 3. The set of the multi-fluid equations and its closure are discussed in Sec. 4. The set of two-dimensional transport equations is derived and it is confirmed that our two-dimensional transport model is consistent with the conventional one-dimensional neoclassical transport model in Sec. 5. In Sec. 6, the set of the electromagnetic equations is derived from Maxwell's equations. In Sec. 7, the procedure for coupling a 2D transport solver with a 2D equilibrium solver is discussed. Summary and discussion are given in Sec. 8.

## 2. Assumptions and Coordinate System

In this paper, we assume axisymmetry of the system and the existence of flux surfaces with two-dimensional equilibrium magnetic field. Based on these assumptions, we employ a magnetic surface coordinate system (MSCS) $(\rho, \chi, \zeta)$ in order to develop a two-dimensional transport model applicable to both the core and the peripheral regions. Here $\rho$ is the radial coordinate label, $\chi$ is the poloidal angle, and $\zeta$ is the toroidal angle. In our MSCS, $\rho$ is defined as the direction perpendicular to the magnetic field $\boldsymbol{B}$ and constructed by the toroidal flux function $\phi \equiv \int_0^\rho \mathrm{d}\rho' \int \mathrm{d}\chi \sqrt{g} B^\zeta / \int \mathrm{d}\chi$, $\chi$ is defined by the normalized length of the field line projected on a constant-$\zeta$ surface and $\zeta$ is defined by the geometrical toroidal angle.

Since the poloidal and toroidal angles are defined independently of the magnetic flux functions, MSCS is a kind of the non-flux coordinate system and is applicable even outside the separatrix on which the safety factor $q \equiv \mathrm{d}\phi/\mathrm{d}\psi$ diverges to infinity, where $\psi \equiv \int_0^\rho \mathrm{d}\rho' \sqrt{g} B^\chi$ is the poloidal flux function. The axisymmetric magnetic field $\boldsymbol{B}$ can be written by the use of the two flux functions $\psi$ and $I = B_\zeta$ [24],

$$\boldsymbol{B} = \nabla\zeta \times \nabla\psi + I\nabla\zeta. \tag{1}$$

The contravariant basis vectors ($\boldsymbol{e}^{\xi_i} \equiv \nabla\xi_i$) for the MSCS ($\xi_i = \rho, \chi, \zeta$) are $\boldsymbol{e}^\rho \equiv \nabla\rho$, $\boldsymbol{e}^\chi \equiv \nabla\chi$, $\boldsymbol{e}^\zeta \equiv \nabla\zeta$. The covariant basis vectors ( $\boldsymbol{e}_{\xi_i} \equiv \partial\boldsymbol{x}/\partial\xi_i$) are $\boldsymbol{e}_\rho \equiv \sqrt{g}\nabla\chi\times\nabla\zeta$, $\boldsymbol{e}_\chi \equiv \sqrt{g}\nabla\zeta \times \nabla\rho$, $\boldsymbol{e}_\zeta \equiv \sqrt{g}\nabla\rho \times \nabla\chi$. The Jacobian is $\sqrt{g}^{-1} \equiv \nabla\rho \cdot \nabla\chi \times \nabla\zeta$. Since the geometrical toroidal angle is employed, the constant-$\zeta$ surface is orthogonal to both the constant-$\rho$ and -$\chi$ surfaces so that $\boldsymbol{e}_\zeta$ and $\boldsymbol{e}^\zeta$ are parallel to one another,

$$\boldsymbol{e}_\zeta = R^2\nabla\zeta = R^2\boldsymbol{e}^\zeta, \quad B_\zeta = B^\zeta R^2, \tag{2}$$

where $R$ is the major radius.

In this paper, the time evolution of the direction of the magnetic field and that of the metric tensor are neglected by assuming the slow change of magnetic flux surface. This assumption will be satisfied in most of phenomena with transport time scale, while it is not satisfied in rapid phenomena with Alfvén time scale.

The relation between the time derivatives in a fixed laboratory frame and in a moving magnetic surface frame can be expressed with a drift velocity of the flux surface $\boldsymbol{u}_g$ as

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial}{\partial t}\right|_{\rho,\chi,\zeta} - \boldsymbol{u}_g \cdot \nabla, \tag{3}$$

where the subscript $\boldsymbol{x}$ indicates the time derivative in the laboratory frame and the subscript $\rho, \chi, \zeta$ in the magnetic surface frame. In the following discussion, the latter subscript is dropped for simplicity. The drift velocity of the magnetic surface is defined by

$$\boldsymbol{u}_g \equiv - \left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{x}} \boldsymbol{e}_\rho - \left.\frac{\partial\chi}{\partial t}\right|_{\boldsymbol{x}} \boldsymbol{e}_\chi = u_g^\rho\boldsymbol{e}_\rho + u_g^\chi\boldsymbol{e}_\chi, \tag{4}$$

and its actual expression depends on the definitions of $\rho$ and $\chi$. From the conservation of volume, Eq. (3) can be transformed as

$$\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}(\sqrt{g}f) - \nabla \cdot \left(\boldsymbol{u}_g f\right). \tag{5}$$

## 3. Small Gyro-Radius Ordering

In this paper, we employ the small gyro-radius ordering in order to formulate the two-dimensional transport model. In this ordering, a small expansion parameter $\delta_a$ for particle species $a$

$$\delta_a \equiv \frac{\varrho_a}{L_\perp} \ll 1, \tag{6}$$

is introduced, where $\varrho_a \equiv v_{Ta}/\omega_{ca}$ is the Larmor radius, $L_\perp$ is the macroscopic characteristic length in the perpendicular direction, $v_{Ta} \equiv \sqrt{2T_a/m_a}$ is the thermal velocity, $\omega_{ca} \equiv |e_a|B/m_a$ is the cyclotron frequency, $e_a$ is the charge and $m_a$ is the mass. Since $\delta_i \sim \sqrt{m_i/m_e}\delta_e$ in general, we consider $\delta \sim \delta_i$ as the most severe restriction for the small gyro-radius ordering.

# 4. Multi-Fluid Equations

We consider the multi-fluid equations which describe the time evolution of macroscopic quantities, such as the particle density $n_a$, the momentum $m_a n_a \boldsymbol{u}_a$, the pressure $p_a$ and the total heat flux $\boldsymbol{Q}_a$, derived from the kinetic equation for each plasma species,

$$\left.\frac{\partial f_a}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}_a \cdot \nabla f_a + \frac{e_a}{m_a}(\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \frac{\partial f_a}{\partial \boldsymbol{v}} = C(f_a) + D_{\mathrm{QL}}(f_a) + S(f_a), \quad (7)$$

where $f_a$ is the distribution function in six-dimensional phase space, $\boldsymbol{v}_a$ is the particle velocity, $C$ is the collision operator, $D_{\mathrm{QL}}$ represents the quasi-linear interaction with waves, and $S$ is the kinetic source. The multi-fluid equations are obtained by taking velocity moments $(1, m\boldsymbol{v}, mv^2/2, mv^2\boldsymbol{v}/2)$ of the kinetic equation as [25]

$$\left.\frac{\partial n_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot (n_a \boldsymbol{u}_a) = S_{na}, \quad (8)$$

$$\left.\frac{\partial}{\partial t}(m_a n_a \boldsymbol{u}_a)\right|_{\boldsymbol{x}} + \nabla \cdot \overleftrightarrow{P}_a = e_a n_a (\boldsymbol{E} + \boldsymbol{u}_a \times \boldsymbol{B}) + \boldsymbol{F}_a + \boldsymbol{F}_a^{\mathrm{QL}} + \boldsymbol{S}_{ma}, \quad (9)$$

$$\left.\frac{\partial}{\partial t}\left(\frac{3}{2}p_a\right)\right|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{Q}_a - \frac{1}{2}m_a n_a u_a^2 \boldsymbol{u}_a\right) = \boldsymbol{u}_a \cdot \nabla p_a + \boldsymbol{u}_a \cdot \nabla \cdot \overleftrightarrow{\pi}_a + Q_{\Delta a} + S_{pa}, \quad (10)$$

$$\left.\frac{\partial \boldsymbol{Q}_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \overleftrightarrow{R}_a = \frac{e_a}{m_a}\left\{\frac{5}{2}p_a \boldsymbol{E} + \overleftrightarrow{\pi}_a \cdot \boldsymbol{E} + \boldsymbol{Q}_a \times \boldsymbol{B}\right\} + \boldsymbol{G}_a + \boldsymbol{G}_a^{\mathrm{QL}} + \boldsymbol{S}_{qa}. \quad (11)$$

The energy transport equation for internal energy (10) is employed instead of that for total energy. In the multi-fluid equations (8)-(11), $S_{na}$ is the particle source, $\overleftrightarrow{P}_a$ is the total stress tensor, $\boldsymbol{F}_a$ is the friction force, $\boldsymbol{F}_a^{\mathrm{QL}}$ represents the interaction with waves, $\boldsymbol{S}_{ma}$ is the momentum source, $\boldsymbol{Q}_a$ is the total heat flux, $Q_{\Delta a}$ is the energy equipartition [26], $S_{pa}$ is the internal energy source, $\overleftrightarrow{R}_a$ is the energy-weighted (EW) total stress tensor, $\boldsymbol{G}_a$ is the EW friction force, $\boldsymbol{G}_a^{\mathrm{QL}}$ is the EW interaction with waves, and $\boldsymbol{S}_{qa}$ is the total heat flux source. The quantities $\overleftrightarrow{P}_a$, $\boldsymbol{Q}_a$, $Q_{\Delta a}$, $S_{pa}$, $\overleftrightarrow{R}_a$, $\boldsymbol{G}_a$ are defined by

$$\overleftrightarrow{P}_a \equiv p_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a + m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a, \quad (12)$$

$$\boldsymbol{Q}_a \equiv \boldsymbol{q}_a + \frac{5}{2}p_a \boldsymbol{u}_a + \overleftrightarrow{\pi}_a \cdot \boldsymbol{u}_a + \frac{1}{2}m_a n_a u_a^2 \boldsymbol{u}_a, \quad (13)$$

$$Q_{\Delta a} \equiv \sum_b \frac{3}{2}n_a \frac{T_b - T_a}{\tau_{ab}}, \quad (14)$$

$$S_{pa} \equiv S_{Ea} - \boldsymbol{u}_a \cdot \boldsymbol{S}_{ma} + \frac{1}{2}m_a u_a^2 S_{na}, \quad (15)$$

$$\overleftrightarrow{R}_a \equiv \frac{5}{2}\frac{T_a}{m_a}p_a \overleftrightarrow{I} + \overleftrightarrow{r}_a + [[\boldsymbol{Q}_a \boldsymbol{u}_a]] - \frac{3}{2}p_a \boldsymbol{u}_a \boldsymbol{u}_a, \quad (16)$$

$$\overleftrightarrow{r}_a \equiv \frac{T_a}{m_a}\left(\frac{5}{2}\overleftrightarrow{\pi}_a + \overleftrightarrow{\theta}_a\right), \quad (17)$$

$$\boldsymbol{G}_a \equiv \frac{T_a}{m_a}\left(\frac{5}{2}\boldsymbol{F}_a + \boldsymbol{H}_a\right), \quad (18)$$

where $\overleftrightarrow{\pi}_a$ is the viscosity tensor, $S_{Ea}$ is the total energy source, $\overleftrightarrow{\theta}_a$ is the heat viscosity tensor, and $\boldsymbol{H}_a$ is the heat friction force. The double square brackets $[[\cdots]]$ is defined as

$$[[\boldsymbol{fg}]] \equiv \boldsymbol{fg} + \boldsymbol{gf}. \quad (19)$$

In Eq. (14), $\tau_{ab}$ is the heat exchange time defined by

$$\tau_{ab} \equiv \frac{3\sqrt{2}\pi^{3/2}\varepsilon_0^2 m_a m_b}{n_b e^4 Z_a^2 Z_b^2 \ln \Lambda}\left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{3/2}, \quad (20)$$

where $\ln \Lambda$ is the Coulomb logarithm.

The turbulent transport is induced by the interaction with low-frequency fluctuations. In the present framework, the quasi-linear term in the kinetic equation generates the force $F_a^{\mathrm{QL}}$ and this force induces particle and heat flux in the perpendicular direction. In the case of the electrostatic fluctuation, the poloidal force acting on electrons can be expresses in the toroidal coordinate $(r, \theta, \phi)$ as [27, 28]

$$F_{e\theta}^{\mathrm{QL}} = eB_\phi n_e D_e\left[-\frac{1}{n_e}\frac{\partial n_e}{\partial r} + \frac{e}{T_e}E_r - \left\langle\frac{\omega}{m}\right\rangle_e r\frac{eB_\phi}{T_e} - \left(\frac{\mu_e}{D_e} - \frac{1}{2}\right)\frac{1}{T_e}\frac{\partial T_e}{\partial r}\right], \quad (21)$$

where $\omega$ and $m$ are the mode frequency and poloidal mode number respectively, and $\langle\omega/m\rangle$ denotes the spectrum average of the phase velocity in the poloidal direction. In the above expression, we have assumed a symmetric wave spectrum with respect to $k_\parallel$ and weak velocity shear. The factor $D_e$ is proportional to the square of the wave amplitude and corresponds to the ordinary diffusion coefficient. If the momentum is conserved between charged particles, the particle flux is intrinsically ambipolar. This particle transport model has been successfully implemented in the TASK/TX code [7]. The momentum and heat flux can be similarly implemented. The parallel component of the turbulence-induced force is neglected for simplicity, since the neoclassical term is considered to be dominant in the parallel direction.

In Eq. (15), the expression of internal energy source $S_{pa}$ differs from the ordinary expression $S_{pa} \equiv S_{Ea} - \frac{1}{2}m_a u_a^2 S_{na}$, since $S_{pa}$ in Eq. (15) includes contributions from not only particle source $S_{na}$ but also momentum source $\boldsymbol{S}_{ma}$. Note that the EW Lorentz force which is the first term in RHS of Eq. (11) and the EW total stress tensor (16) have been simplified to the extent that they keep consistency with the neoclassical theory.

The viscosity tensor $\overleftrightarrow{\pi}_a$, the heat viscosity tensor $\overleftrightarrow{\theta}_a$, the friction force $\boldsymbol{F}_a$, and the heat friction force $\boldsymbol{H}_a$ must be modeled in order to complete the multi-fluid equations. According to the moment approach, the lowest order friction force $\boldsymbol{F}_a$ and heat friction force $\boldsymbol{H}_a$ can be expressed in terms of flows

$$\boldsymbol{F}_a = \sum_b \left( l_{11}^{ab} \boldsymbol{u}_b - l_{12}^{ab} \frac{2\boldsymbol{q}_b}{5p_b} \right), \tag{22}$$

$$\boldsymbol{H}_a = \sum_b \left( -l_{21}^{ab} \boldsymbol{u}_b + l_{22}^{ab} \frac{2\boldsymbol{q}_b}{5p_b} \right), \tag{23}$$

where the coefficients $l_{ij}^{ab}$ can be expressed in terms of the Braginskii's matrix elements of the collision operator [12].

In the lowest order of the drift ordering $O(\delta)$, the viscosity tensor $\overleftrightarrow{\pi}_a$ and the heat viscosity tensor $\overleftrightarrow{\theta}_a$ are in the CGL form as

$$\overleftrightarrow{\pi}_a = \pi_{\|a} \left( \boldsymbol{e}_\| \boldsymbol{e}_\| - \frac{1}{3} \overleftrightarrow{I} \right) + O(\delta^2), \tag{24}$$

$$\overleftrightarrow{\theta}_a = \theta_{\|a} \left( \boldsymbol{e}_\| \boldsymbol{e}_\| - \frac{1}{3} \overleftrightarrow{I} \right) + O(\delta^2), \tag{25}$$

where $\boldsymbol{e}_\| \equiv \boldsymbol{B}/B$ is the unit vector in the parallel direction. In this paper, we define the parallel viscosities $\pi_{\|a}$ and $\theta_{\|a}$ in terms of the neoclassical parallel viscosity coefficients $\mu_{ai}$ and the parallel-parallel components of the rate-of-strain tensors $W_{zz}^{ua}$ and $W_{zz}^{qa}$ as

$$\begin{bmatrix} \pi_{\|a} \\ \theta_{\|a} \end{bmatrix} = -\frac{3}{2} \begin{bmatrix} \mu_{a1} & \mu_{a2} \\ \mu_{a2} & \mu_{a3} \end{bmatrix} \begin{bmatrix} W_{zz}^{ua} \\ W_{zz}^{qa} \end{bmatrix}, \tag{26}$$

where

$$W_{zz}^{ua} = 2 \left( \nabla_\| u_{a\|} - \boldsymbol{u}_a \cdot \boldsymbol{\kappa} \right), \tag{27}$$

$$W_{zz}^{qa} = 2 \left( \nabla_\| \left( \frac{2q_{a\|}}{5p_a} \right) - \frac{2\boldsymbol{q}_a}{5p_a} \cdot \boldsymbol{\kappa} \right). \tag{28}$$

In Eqs. (27) and (28), the incompressibility of flows, $\nabla \cdot \boldsymbol{u}_a = 0$ and $\nabla \cdot (2\boldsymbol{q}_a/5p_a) = 0$, have been assumed for simplicity and $\boldsymbol{\kappa} = \boldsymbol{e}_\| \cdot \nabla \boldsymbol{e}_\|$ is the magnetic curvature. It is easily shown that Eq. (26) is equivalent to the Hirshman-type parallel viscosities inside the LCFS in the sense of the flux averaged viscous forces $\left\langle \boldsymbol{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_a \right\rangle$ and $\left\langle \boldsymbol{B} \cdot \nabla \cdot \overleftrightarrow{\theta}_a \right\rangle$ and also equivalent to the Braginskii-type parallel viscosity outside the LCFS [29].

In the above discussion, the neoclassical parallel coefficients $\mu_{ai}$ obtained from the bounce-averaged drift kinetic equation are assumed, which means that $\mu_{ai}$ in Eq. (26) are flux functions and lose their poloidal dependence. In the core region, the equilibrium return flows are formed and transport is essentially reduced to one-dimensional problem. Therefore the poloidal dependence of $\mu_{ai}$ is assumed to be small enough to be negligible. In the edge region where the plasma is weakly collisional, $\mu_{ai}$ should have weak poloidal dependence. We assume, however, that the effect of the poloidal non-uniformity of the plasma density and temperature on the viscosity is small and use the bounce averaged $\mu_{ai}$ as an approximation. In the peripheral region where the plasma is collisional, Eq. (26) is reduce to the Braginskii's expression [12, 29] so that Eq. (26) recovers its poloidal dependence.

# 5. Derivation of Two-Dimensional Transport Equations

In this section, we derive the two-dimensional transport modeling equations composed of the equations for particle density, momentum in the three direction (radial, parallel and toroidal), internal energy, and total heat flux in the three direction (radial, parallel and toroidal) for each species, and Maxwell's equations for electromagnetic field. Since the toroidal symmetry is assumed, the spatial variation of quantities are two-dimensional, in the radial and poloidal directions. Since the parallel flows have great influence on tokamak transport, we consider three components of vector quantities, $(\rho, \|, \zeta)$ in the radial, the parallel to the field line, and the toroidal, rather than those of MSCS $(\rho, \chi, \zeta)$.

## 5.1 Equation for particle density

In this paper the equation of continuity (8) is employed as the equation for particle density,

$$\left.\frac{\partial n_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot (n_a \boldsymbol{u}_a) = S_{na}. \tag{29}$$

## 5.2 Equation of motion in the parallel direction

We formulate the evolution equation for the parallel momentum by taking a scalar product of the equation of motion (9) and $\boldsymbol{B}$:

$$\begin{aligned} \frac{\partial}{\partial t} \left. (m_a n_a u_{a\|} B) \right|_{\boldsymbol{x}} &+ \boldsymbol{B} \cdot \nabla \cdot (m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a) \\ &+ B \nabla_\| p_a + \boldsymbol{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_a \\ &= e_a n_a B E_\| + F_{a\|} B + F_{a\|}^{\mathrm{QL}} B + S_{ma\|} B. \end{aligned} \tag{30}$$

The time derivative term in Eq. (30) is reduced, since the time variation of magnetic field is much slower than that of momentum, where we have evaluated $\boldsymbol{u}_a \sim O(\delta)$, $\partial \boldsymbol{B}/\partial t|_{\boldsymbol{x}} \sim O(\delta^2)$ and $\partial (m_a n_a u_{a\|} B)/\partial t|_{\boldsymbol{x}} \sim O(\delta^2)$. Though the inertial force driven by the drift of the flux surfaces $\boldsymbol{u}_g$ included in the time derivative term in Eq. (30) is $O(\delta^3)$, we retain it from the aspect of volume conservation.

Next, we evaluate the inertial force in the parallel direction. To obtain a simple expression, we split the flow velocity into the parallel and the perpendicular components, $\boldsymbol{u}_a = \boldsymbol{u}_{a\|} + \boldsymbol{u}_{a\perp}$. The inertial stress tensor $m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a$ now is split into 4 terms and we keep terms up to $O(\delta^2)$ in our transport model:

$$m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a = m_a n_a \boldsymbol{u}_{a\|} \boldsymbol{u}_{a\|} + O(\delta^3). \tag{31}$$

Therefore the inertial force in the parallel direction $F_{ua\|}^{\mathrm{ine}}$ is rewritten in a simple form:

$$\begin{aligned} F_{ua\|}^{\mathrm{ine}} B &= \boldsymbol{B} \cdot \nabla \cdot (m_a n_a \boldsymbol{u}_{a\|} \boldsymbol{u}_{a\|}) \\ &= B \nabla_\| (m_a n_a u_{a\|} u_{a\|}) - m_a n_a u_{a\|} u_{a\|} \nabla_\| B, \end{aligned} \tag{32}$$

where we have used the following relation

$$\boldsymbol{B} \cdot \nabla \cdot (f \boldsymbol{e}_\| \boldsymbol{e}_\|) = B \nabla_\| f - f \nabla_\| B. \tag{33}$$

The viscous force in the parallel direction $F^{\rm vis}_{ua\|}$ can be written as

$$F^{\rm vis}_{ua\|}B = \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}_a = -\pi_{\|a}\nabla_\| B + \frac{2}{3}B\nabla_\|\pi_{\|a}, \qquad (34)$$

since the parallel viscosity tensor is in the CGL form.

From Eq. (30), the force due to the pressure gradient in the parallel direction $F^{\nabla p}_{ua\|}$, the Lorentz force in the parallel direction $F^{\rm Lor}_{ua\|}$ and the friction force in the parallel direction $F^{\rm fri}_{ua\|}$ can be written respectively as

$$F^{\nabla p}_{ua\|}B = B\nabla_\| p_a, \qquad (35)$$

$$F^{\rm Lor}_{ua\|}B = e_a n_a E_\| B, \qquad (36)$$

$$F^{\rm fri}_{ua\|}B = \sum_b \left(l^{ab}_{11}u_{b\|} - l^{ab}_{12}\frac{2q_{b\|}}{5p_b}\right)B. \qquad (37)$$

Therefore the equation for the parallel momentum is obtained as

$$\frac{\partial}{\partial t}\left(m_a n_a u_{a\|}B\right)\bigg|_{\boldsymbol{x}} + F^{\rm ine}_{ua\|}B + F^{\nabla p}_{ua\|}B + F^{\rm vis}_{ua\|}B = F^{\rm Lor}_{ua\|}B + F^{\rm fri}_{ua\|}B + F^{\rm QL}_{a\|}B + S_{ma\|}B. \quad (38)$$

### 5.3 Equation of motion in the toroidal direction

Taking the scalar product of the equation of motion (9) and the covariant toroidal basis $\boldsymbol{e}_\zeta$, we obtain

$$\frac{\partial}{\partial t}\left(m_a n_a u_{a\zeta}\right)\bigg|_{\boldsymbol{x}} + \nabla\cdot\left(\boldsymbol{e}_\zeta\cdot\overleftrightarrow{P}_a\right) = e_a n_a\left(E_\zeta + \psi' u^\rho_a\right) + F_{a\zeta} + F^{\rm QL}_{a\zeta} + S_{ma\zeta}, \qquad (39)$$

where $\zeta$ is defined geometrically so that its time derivative is identically zero and $\psi'$ indicates the derivative of $\psi$ with respect to $\rho$.

Since the total stress tensor $\overleftrightarrow{P}_a$ is symmetric, the following useful identity of the second-rank symmetric tensor $\overleftrightarrow{S}$ has been used in taking the toroidal projection of total stress $\boldsymbol{e}_\zeta\cdot\nabla\cdot\overleftrightarrow{P}_a$:

$$\boldsymbol{e}_\zeta\cdot\nabla\cdot\overleftrightarrow{S} = \nabla\cdot\left(\boldsymbol{e}_\zeta\cdot\overleftrightarrow{S}\right). \qquad (40)$$

The inertial force in the toroidal direction $F^{\rm ine}_{ua\zeta}$ and the viscous force in the toroidal direction $F^{\rm vis}_{ua\zeta}$ therefore can be expressed as

$$F^{\rm ine}_{ua\zeta} = \nabla\cdot\left(m_a n_a u_{a\zeta}\boldsymbol{u}_a\right), \qquad (41)$$

$$F^{\rm vis}_{ua\zeta} = B\nabla_\|\left(\frac{I}{B^2}\pi_{\|a}\right). \qquad (42)$$

Note that the parallel viscous force in the toroidal direction may not vanish in two-dimensional transport modeling in contrast to the traditional one-dimensional transport modeling. It is easily confirmed that the flux-surface-averaged value of Eq. (42) vanishes as $\langle\boldsymbol{B}\cdot\nabla f\rangle = 0$, which is consistent with the one-dimensional transport theory.

The Lorentz force in the toroidal direction $F^{\rm Lor}_{ua\zeta}$ and the friction force in the toroidal direction $F^{\rm fri}_{ua\zeta}$ can be written as

$$F^{\rm Lor}_{ua\zeta} = e_a n_a E_\zeta + e_a n_a\psi' u^\rho_a, \qquad (43)$$

$$F^{\rm fri}_{ua\zeta} = \sum_b\left(l^{ab}_{11}u_{b\zeta} - l^{ab}_{12}\frac{2q_{b\zeta}}{5p_b}\right). \qquad (44)$$

Therefore, the equation for the toroidal momentum is obtained as follows:

$$\frac{\partial}{\partial t}\left(m_a n_a u_{a\zeta}\right)\bigg|_{\boldsymbol{x}} + F^{\rm ine}_{ua\zeta} + F^{\rm vis}_{ua\zeta} = F^{\rm Lor}_{ua\zeta} + F^{\rm fri}_{ua\zeta} + F^{\rm QL}_{a\zeta} + S_{ma\zeta}. \qquad (45)$$

### 5.4 Equation of radial force balance

Since the time derivative of the radial momentum is $O(\delta^3)$ and small enough to be negligible, we assume the lowest order $O(1)$ force balance in the radial direction for simplicity:

$$\nabla\rho\cdot\nabla p_a = e_a n_a E^\rho + \nabla\rho\cdot\left(e_a n_a\boldsymbol{u}_a\times\boldsymbol{B}\right). \qquad (46)$$

From Eq. (46), the force due to the pressure gradient in the radial direction $F^{\nabla p\ \rho}$ and the Lorentz force in the radial direction $F^{\rm Lor\ \rho}_{ua}$ can be written as

$$F^{\nabla p\ \rho}_{ua} = g^{\rho\rho}\frac{\partial p_a}{\partial\rho} + g^{\rho\chi}\frac{\partial p_a}{\partial\chi}, \qquad (47)$$

$$F^{\rm Lor\ \rho}_{ua} = e_a n_a E^\rho_a + e_a\frac{IB}{\psi'}n_a u_{a\|} - e_a\frac{B^2}{\psi'}n_a u_{a\zeta}, \qquad (48)$$

where the following relation have been used in Eq. (48):

$$\nabla\rho\cdot(\boldsymbol{f}\times\boldsymbol{B}) = \frac{IB}{\psi'}f_\| - \frac{B^2}{\psi'}f_\zeta. \qquad (49)$$

Therefore, we obtain the equation of the force balance in the radial direction:

$$F^{\nabla p\ \rho}_{ua} = F^{\rm Lor\ \rho}_{ua}. \qquad (50)$$

### 5.5 Equation for energy transport

The energy transport equation for internal energy does not change from Eq. (10), since the equation for total heat flux $\boldsymbol{Q}_a$ is solved simultaneously. We substitute Eq. (13) into Eq. (10), however, in order to evaluate the terms in Eq. (10) in terms of $\delta$

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_a\right)\bigg|_{\boldsymbol{x}} + \nabla\cdot\left(\boldsymbol{q}_a + \frac{5}{2}p_a\boldsymbol{u}_a + \overleftrightarrow{\pi}_a\cdot\boldsymbol{u}_a\right) = \boldsymbol{u}_a\cdot\nabla p_a + \boldsymbol{u}_a\cdot\nabla\cdot\overleftrightarrow{\pi}_a + S_{pa}. \qquad (51)$$

Moreover, Eq. (51) can be transformed to the expression for the adiabatic entropy $\sqrt{g}^{5/3}p_a$. All terms in Eq. (51) are $O(\delta^2)$ in the equilibrium state.

The viscous heating term by the parallel viscous force $Q^{\rm vis}_a \equiv \boldsymbol{u}_a\cdot\nabla\cdot\overleftrightarrow{\pi}_a$ can be written as

$$Q^{\rm vis}_a = B\nabla_\|\left(\frac{u_{a\|}\pi_{\|a}}{B}\right) - \pi_{\|a}\left(\nabla_\| u_{a\|} - \boldsymbol{u}_a\cdot\boldsymbol{\kappa}\right) - \frac{1}{3}\boldsymbol{u}_a\cdot\nabla\pi_{\|a}, \qquad (52)$$

since $\overleftrightarrow{\pi}_a$ is in the CGL form. Now we will show that Eq. (52) is consistent with the result of one-dimensional modeling. Substituting the equilibrium return flows,

$$\bar{\boldsymbol{u}}_a \equiv \omega_{ua} R^2 \nabla\zeta + L_{ua}\boldsymbol{B}, \tag{53}$$

$$\bar{\boldsymbol{q}}_a \equiv \omega_{qa} R^2 \nabla\zeta + L_{qa}\boldsymbol{B}, \tag{54}$$

into Eq. (52) and averaging it over the flux surfaces, we obtain

$$\left\langle Q_a^{\text{vis}} \right\rangle = L_{ua} \left\langle \boldsymbol{B} \cdot \nabla \cdot \overleftrightarrow{\pi}_a \right\rangle, \tag{55}$$

where $\omega_{ua}$ and $\omega_{qa}$ are the toroidal angular frequencies and $L_{ua}$ and $L_{qa}$ are quantities related to the equilibrium poloidal flows. We have used $\langle \boldsymbol{B} \cdot \nabla f \rangle = 0$ in the derivation of Eq. (55). Equation (55) is consistent with the viscous heating term in the one-dimensional transport modeling [30].

Therefore, the equation for internal energy is

$$\frac{3}{2}\left.\frac{\partial p_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{Q}_a - \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a\right) = \boldsymbol{u}_a \cdot \nabla p_a + Q_a^{\text{vis}} + Q_{\Delta a} + S_{pa}. \tag{56}$$

## 5.6 Equations for total heat flux

The equations for total heat flux can be derived by analogy with the derivation of the equation for momentum.

Taking a scalar product of the equation for total heat flux (11) and $\boldsymbol{B}$, we obtain the equation for total heat flux in the parallel direction

$$\left.\frac{\partial}{\partial t}(Q_{a\|}B)\right|_{\boldsymbol{x}} + F_{qa\|}^{\text{ine}} B + F_{qa\|}^{\nabla p} B + F_{qa\|}^{\text{vis}} B = F_{qa\|}^{\text{Lor}} B + F_{qa\|}^{\text{fri}} B + G_{a\|}^{\text{QL}} B + S_{qa\|} B, \tag{57}$$

where $F_{qa\|}^{\text{ine}}$, $F_{qa\|}^{\nabla p}$, $F_{qa\|}^{\text{vis}}$, $F_{qa\|}^{\text{Lor}}$ and $F_{qa\|}^{\text{fri}}$ are the EW inertial force, the EW force due to the EW pressure gradient, the EW viscous force, the EW Lorentz force and the EW friction force in the parallel direction respectively and defined as

$$F_{qa\|}^{\text{ine}} B \equiv B\nabla_\| \left([[\boldsymbol{Q}_a \boldsymbol{u}_a]]_{zz} - \frac{3}{2} p_a u_{a\|} u_{a\|}\right) - \left([[\boldsymbol{Q}_a \boldsymbol{u}_a]]_{zz} - \frac{3}{2} p_a u_{a\|} u_{a\|}\right) \nabla_\| B, \tag{58}$$

$$F_{qa\|}^{\nabla p} B \equiv B\nabla_\| \left(\frac{5T_a}{2m_a} p_a\right), \tag{59}$$

$$F_{qa\|}^{\text{vis}} B \equiv -r_{\|a} \nabla_\| B + \frac{2}{3} B \nabla_\| r_{\|a}, \tag{60}$$

$$F_{qa\|}^{\text{Lor}} B \equiv \frac{e_a}{m_a} \left(\frac{5}{2} p_a + \frac{2}{3} \pi_{a\|}\right) E_\| B, \tag{61}$$

$$F_{qa\|}^{\text{fri}} B \equiv \frac{5T_a}{2m_a} \sum_b \left(l_{11}^{ab} u_{a\|} - l_{12}^{ab} \frac{2q_{b\|}}{5p_b}\right) B + \frac{T_a}{m_a} \sum_b \left(-l_{21}^{ab} u_{a\|} + l_{22}^{ab} \frac{2q_{b\|}}{5p_b}\right) B, \tag{62}$$

where $r_{\|a}$ is the EW parallel viscosity,

$$r_{\|a} \equiv \frac{T_a}{m_a} \left(\frac{5}{2} \pi_{\|a} + \theta_{\|a}\right). \tag{63}$$

Taking a scalar product of the equation for total heat flux (11) and $\boldsymbol{e}_\zeta$, we obtain the equation for total heat flux in the toroidal direction,

$$\left.\frac{\partial Q_{a\zeta}}{\partial t}\right|_{\boldsymbol{x}} + F_{qa\zeta}^{\text{ine}} + F_{qa\zeta}^{\text{vis}} = F_{qa\zeta}^{\text{Lor}} + F_{qa\zeta}^{\text{fri}} + G_{a\zeta}^{\text{QL}} + S_{qa\zeta}, \tag{64}$$

where $F_{qa\zeta}^{\text{ine}}$, $F_{qa\zeta}^{\text{vis}}$, $F_{qa\zeta}^{\text{Lor}}$ and $F_{qa\zeta}^{\text{fri}}$ are the EW inertial force, the EW viscous force, the EW Lorentz force and the EW friction force in the toroidal direction respectively and defined as

$$F_{qa\zeta}^{\text{ine}} \equiv \nabla \cdot \left(\boldsymbol{e}_\zeta \cdot [[\boldsymbol{Q}_a \boldsymbol{u}_a]] - \frac{3}{2} p_a u_{a\zeta} \boldsymbol{u}_a\right), \tag{65}$$

$$F_{qa\zeta}^{\text{vis}} \equiv B\nabla_\| \left(\frac{I}{B^2} r_{a\|}\right), \tag{66}$$

$$F_{qa\zeta}^{\text{Lor}} \equiv \frac{e_a}{m_a} \left[\left(\frac{5}{2} p_a - \frac{1}{3} \pi_{a\|}\right) E_\zeta + \frac{I}{B} \pi_{a\|} E_\| + \psi' Q_a^\rho\right], \tag{67}$$

$$F_{qa\zeta}^{\text{fri}} \equiv \frac{5T_a}{2m_a} \sum_b \left(l_{11}^{ab} u_{a\zeta} - l_{12}^{ab} \frac{2q_{b\zeta}}{5p_b}\right) + \frac{T_a}{m_a} \sum_b \left(-l_{21}^{ab} u_{a\zeta} + l_{22}^{ab} \frac{2q_{b\zeta}}{5p_b}\right). \tag{68}$$

The equation for total heat flux in the radial direction in the lowest order is given by

$$F_{qa}^{\nabla p\ \rho} = F_{qa}^{\text{Lor}\ \rho}, \tag{69}$$

where $F_{qa}^{\nabla p\ \rho}$ and $F_{qa}^{\text{Lor}\ \rho}$ are the force due to the EW pressure gradient and the EW Lorentz force in the radial direction respectively and defined as

$$F_{qa}^{\nabla p\ \rho} \equiv g^{\rho\rho} \frac{\partial}{\partial \rho} \left(\frac{5T_a}{2m_a} p_a\right) + g^{\rho\chi} \frac{\partial}{\partial \chi} \left(\frac{5T_a}{2m_a} p_a\right), \tag{70}$$

$$F_{qa}^{\text{Lor}\ \rho} \equiv \frac{5}{2} \frac{T_a}{m_a} e_a n_a E^\rho + \frac{e_a}{m_a} \frac{IB}{\psi'} Q_{a\|} - \frac{e_a}{m_a} \frac{B^2}{\psi'} Q_{a\zeta}. \tag{71}$$

## 5.7 Consistency with the conventional neoclassical transport theory

We will show that our two-dimensional transport model is consistent with the ordinary flux-surface-averaged neoclassical transport theory [11, 12]. Assuming the equilibrium state inside of the LCFS and the force balance up to $O(\delta)$ in Eq. (38) and averaging on the flux surfaces, we obtain

$$\left\langle F_{ua\|}^{\nabla p} B \right\rangle + \left\langle F_{ua\|}^{\text{vis}} B \right\rangle = \left\langle F_{ua\|}^{\text{Lor}} B \right\rangle + \left\langle F_{ua\|}^{\text{fri}} B \right\rangle. \tag{72}$$

Substituting Eqs. (53) and (54) into Eq. (72), we obtain

$$\left\langle 3(\nabla_\parallel B)^2 \right\rangle \left( \mu_{a1} L_{ua} + \mu_{a2} \frac{2L_{qa}}{5p_a} \right) = \sum_b \left( l_{11}^{ab} \langle u_{b\parallel} B \rangle - l_{12}^{ab} \langle q_{b\parallel} B \rangle \right) + e_a n_a \langle E_\parallel B \rangle, \tag{73}$$

where we have used $\langle \boldsymbol{B} \cdot \nabla f \rangle = 0$. The flux-surface-averaged parallel force balance up to $O(\delta)$ in Eq. (57) also becomes

$$\left\langle F_{qa}^{\nabla p} B \right\rangle + \left\langle F_{qa\parallel}^{\mathrm{vis}} B \right\rangle = \left\langle F_{qa\parallel}^{\mathrm{Lor}} B \right\rangle + \left\langle F_{qa\parallel}^{\mathrm{fri}} B \right\rangle, \tag{74}$$

and we obtain

$$\left\langle 3(\nabla_\parallel B)^2 \right\rangle \left( \mu_{a2} L_{ua} + \mu_{a3} \frac{2L_{qa}}{5p_a} \right) = \sum_b \left( -l_{21}^{ab} \langle u_{b\parallel} B \rangle + l_{22}^{ab} \langle q_{b\parallel} B \rangle \right), \tag{75}$$

where we have used $\langle \boldsymbol{B} \cdot \nabla f \rangle = 0$ and Eq. (73). The flux-surface-averaged parallel flows $\langle u_{a\parallel} B \rangle$ and $\langle q_{a\parallel} B \rangle$ are decomposed by the use of Eqs. (53) and (54)

$$\langle u_{a\parallel} B \rangle = V_{1a} B + L_{ua} \left\langle B^2 \right\rangle, \quad V_{1a} \equiv \frac{I}{B} \omega_{ua}, \tag{76}$$

$$\langle q_{a\parallel} B \rangle = V_{2a} B + \frac{2L_{qa}}{5p_a} \left\langle B^2 \right\rangle, \quad V_{2a} \equiv \frac{I}{B} \omega_{qa}. \tag{77}$$

Substituting Eqs. (76) and (77) into Eqs. (73) and (75), the equations for poloidal rotations in the conventional neoclassical theory is obtained as

$$\left\langle 3\left(\nabla_\parallel B\right)^2 \right\rangle \begin{pmatrix} \mu_{1a} & \mu_{2a} \\ \mu_{2a} & \mu_{3a} \end{pmatrix} \begin{pmatrix} L_{ua} \\ \dfrac{2L_{qa}}{5p_a} \end{pmatrix} = \sum_b \begin{pmatrix} l_{11}^{ab} & -l_{12}^{ab} \\ -l_{21}^{ab} & l_{22}^{ab} \end{pmatrix} \begin{pmatrix} V_{1b} B + L_{ub} \left\langle B^2 \right\rangle \\ V_{2b} B + \dfrac{2L_{qb}}{5p_b} \left\langle B^2 \right\rangle \end{pmatrix} + \begin{pmatrix} e_a n_a \langle E_\parallel B \rangle \\ 0 \end{pmatrix}. \tag{78}$$

# 6. Derivation of Electromagnetic Equations

In this section, the electromagnetic equations are derived from Maxwell's equations:

$$\left.\frac{\partial \boldsymbol{B}}{\partial t}\right|_{\boldsymbol{x}} + \nabla \times \boldsymbol{E} = \boldsymbol{0}, \tag{79}$$

$$\frac{1}{c^2} \left.\frac{\partial \boldsymbol{E}}{\partial t}\right|_{\boldsymbol{x}} - \nabla \times \boldsymbol{B} + \mu_0 \boldsymbol{j} = \boldsymbol{0}, \tag{80}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{81}$$

$$\nabla \cdot \boldsymbol{E} = \frac{\sigma}{\varepsilon_0}, \tag{82}$$

where $\sigma$ is the electric charge density. Gauss's law for magnetism (81) has already been taken into account through the expression of the equilibrium magnetic field.

Variables used to describe the electromagnetic field are chosen as follows. For the magnetic field $\boldsymbol{B}$, the contravariant poloidal component $B^\chi (= \psi' \sqrt{g}^{-1})$ and the covariant toroidal component $B_\zeta (= I)$ are suitable for describing the magnetic field $\boldsymbol{B}$ in Eq. (1). For the electric field $\boldsymbol{E}$, the covariant components are suitable for taking a scalar product of $\boldsymbol{E}$ and $\boldsymbol{B}$. We should note that the distinction between the covariant and the contravariant components is not essential in the toroidal direction in MSCS owing to its orthogonality in the toroidal direction. Therefore, the following five variables are employed to describe the evolution of the electromagnetic field, $\psi'$, $I$, $E_\rho$, $E_\chi$ and $E_\zeta$.

Since the existence of magnetic surfaces is assumed, $\psi'$ and $I$ are the flux functions. From Faraday's law, $E_\zeta$ is also the flux function as is shown later. Taking account of the consistency with these properties, we introduce flux-surface-average for some of electromagnetic field equations. This approximation is necessary for the compatibility of the two-dimensional transport analysis with the existence of magnetic surfaces. The validity of this approximation has to be examined a posteriori.

For Faraday's law (79), the contravariant poloidal direction $\nabla \chi$ and the toroidal direction $\nabla \zeta$ are chosen for the direction of projection, since there is no contravariant radial component of the magnetic field in MSCS. For Ampère's law (80), the projection in the parallel direction $\boldsymbol{B}$ and the toroidal direction $\nabla \zeta$ are used owing to the compatibility with the direction of the current density $\boldsymbol{j}$ derived from the equation of motion. Instead of the contravariant radial component of Ampère's law, we solve Gauss's law (82) which is the time integral of the divergence of Ampère's law.

## 6.1 Equations for magnetic field

In this section, we will derive the equations for $\psi'$ and $I$ from Faraday's law (79). Substituting Eq. (1) into Faraday's law (79), we obtain

$$\left.\frac{\partial \psi'}{\partial t}\right|_{\boldsymbol{x}} \nabla \zeta \times \nabla \rho + \left.\frac{\partial I}{\partial t}\right|_{\boldsymbol{x}} \boldsymbol{e}^\zeta + \nabla \times \boldsymbol{E} = \boldsymbol{0}. \tag{83}$$

Taking a scalar product of (83) and $\nabla \chi$, we obtain the equation for $\psi'$

$$\left.\frac{\partial \psi'}{\partial t}\right|_{\boldsymbol{x}} - \frac{\partial E_\zeta}{\partial \rho} = 0. \tag{84}$$

Since $\psi'$ is the flux function, $E_\zeta$ is also the flux function.

Since the covariant toroidal magnetic field $B_\zeta (= I)$ is the flux function, we take a scalar product of (83) and $\boldsymbol{e}_\zeta$ and the $\nabla \times \boldsymbol{E}$ term is averaged over the flux surfaces to obtain

$$\left.\frac{\partial I}{\partial t}\right|_{\boldsymbol{x}} + \left\langle \frac{R^2}{\sqrt{g}} \left( \frac{\partial E_\chi}{\partial \rho} - \frac{\partial E_\rho}{\partial \chi} \right) \right\rangle = 0. \tag{85}$$

### 6.2 Equations for electric field

In this section the equations for the covariant toroidal electric field $E_\zeta$ and the covariant poloidal electric field $E_\chi$ are derived from Ampère's law and the equation for the covariant radial electric field $E_\rho$ from Gauss's law. The rotation of the magnetic field can be expressed as

$$\begin{aligned}\nabla\times\boldsymbol{B} &= \nabla\times(\nabla\zeta\times\nabla\psi+I\nabla\zeta)\\ &= \nabla\cdot(\nabla\psi\nabla\zeta-\nabla\zeta\nabla\psi)+\nabla I\times\nabla\zeta\\ &= \nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right)R^2\nabla\zeta+\nabla I\times\nabla\zeta, \end{aligned}\tag{86}$$

where $\nabla\psi\nabla\zeta-\nabla\zeta\nabla\psi$ is a 2nd-rank antisymmetric tensor and the following tensor identities for any vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ and any second-rank antisymmetric tensor $\overset{\leftrightarrow}{A}$ have been employed:

$$\nabla\times(\boldsymbol{f}\times\boldsymbol{g}) = \nabla\cdot(\boldsymbol{g}\boldsymbol{f}-\boldsymbol{f}\boldsymbol{g}),\tag{87}$$

$$\nabla\cdot\overset{\leftrightarrow}{A} = \sum_{\xi_i=\rho,\chi,\zeta}\nabla\cdot\left(\overset{\leftrightarrow}{A}\cdot\boldsymbol{e}^{\xi_i}\right)\boldsymbol{e}_{\xi_i},\tag{88}$$

Substituting Eq. (86) into Eq. (80), we obtain the equation for the electric field in the axisymmetric system,

$$\frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t}\bigg|_{\boldsymbol{x}}-\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right)R^2\nabla\zeta -\nabla I\times\nabla\zeta+\mu_0\boldsymbol{j}=\boldsymbol{0}.\tag{89}$$

Taking a scalar product of Eq. (89) and $\boldsymbol{e}_\zeta$, we obtain the equation for the covariant toroidal electric field $E_\zeta$,

$$\frac{1}{c^2}\frac{\partial E_\zeta}{\partial t}\bigg|_{\boldsymbol{x}}-R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right)+\mu_0 j_\zeta=0.\tag{90}$$

This equation reduces to the Grad-Shafranov equation in a stationary state. Since $E_\zeta$ is the flux function, we employ the flux-surface-average of the second and the third terms to obtain,

$$\frac{1}{c^2}\frac{\partial E_\zeta}{\partial t}\bigg|_{\boldsymbol{x}}-\left\langle R^2\nabla\cdot\left(\frac{\psi'}{R^2}\nabla\rho\right)\right\rangle+\mu_0\left\langle j_\zeta\right\rangle=0.\tag{91}$$

This equation corresponds to the flux-surface-averaged Grad-Shafranov equation employed in the Flux Conserving Tokamak (FCT) scheme [31–33].

Taking a scalar product of Eq. (89) and $\boldsymbol{B}$, we obtain

$$\frac{1}{c^2}\left(\frac{\psi'}{\sqrt{g}}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}}+I\frac{\partial E^\zeta}{\partial t}\bigg|_{\boldsymbol{x}}\right)-\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right)I +\frac{g^{\rho\rho}}{R^2}\psi'\frac{\mathrm{d}I}{\mathrm{d}\rho}+\mu_0 j_\parallel B=0.\tag{92}$$

Substituting Eq. (90) into Eq. (92), we obtain the equation for the covariant poloidal electric field $E_\chi$,

$$\frac{1}{c^2}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}}+\frac{g_{\chi\chi}}{\sqrt{g}}\frac{\mathrm{d}I}{\mathrm{d}\rho}+\mu_0\frac{\sqrt{g}\left(j_\parallel B-j^\zeta I\right)}{\psi'}=0.\tag{93}$$

Finally the covariant radial electric field $E_\rho$ is obtained by solving Gauss's law,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\rho}\left[\sqrt{g}\left(g^{\rho\rho}E_\rho+g^{\rho\chi}E_\chi\right)\right] +\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(g^{\chi\rho}E_\rho+g^{\chi\chi}E_\chi\right)\right]=\frac{\sigma}{\varepsilon_0}.\tag{94}$$

## 7. Connection between Transport and Equilibrium Solver

In this section we briefly describe the procedure for coupling the transport solver with an equilibrium solver. At the beginning, MSCS is calculated by solving the Grad-Shafranov equation for initial profiles.

At the first step, in the transport solver, the set of transport equations, Eqs. (29), (38), (45), (50), (56), (57), (64) and (69), and the set of electromagnetic equations, Eqs. (84), (85), (91), (93) and (94) are solved simultaneously in MSCS in an implicit way. Since the transport coefficients and the source terms depend on the plasma quantities, densities, temperatures, and flows, this procedure has to be repeated until the solutions are converged.

At the second step, the two-dimensional toroidal current density profile $j_\zeta(\rho,\chi)$ and the toroidal component of the displacement current density profile $j_\zeta^{dc}(\rho)=1/(\mu_0c^2)\partial E_\zeta/\partial t|_{\boldsymbol{x}}$ calculated by the transport solver in MSCS, are converted to the two-dimensional profiles $j_\zeta(R,Z)$ and $j_\zeta^{dc}(R,Z)$ in the cylindrical coordinate system $(R,\varphi,Z)$ and sent to the free-boundary equilibrium solver.

At the third step, in the free-boundary equilibrium solver, Eq. (90) is solved with given $j_\zeta(R,Z)$ and $j_\zeta^{dc}(R,Z)$ to calculate $\psi(R,Z)$,

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\psi}{\partial Z^2}=\frac{\mu_0}{R^2}\left(j_\zeta+j_\zeta^{dc}\right).\tag{95}$$

In this recalculation of the equilibrium magnetic field, we employ the FCT scheme [31–33] in which the toroidal and poloidal fluxes are conserved; therefore $q(\rho)$ is unchanged. The particle density, the momentum, the pressure and the heat flux are also changed adiabatically according to the change of volume. In order to obtain the equilibrium satisfying these constrains, Eqs. (91) and (95) are solved iteratively. In the 1.5D transport modeling, Eq. (91) without the displacement current is solved for fixed $q(\rho)$ and $p(\rho)$ with adiabatic constraint to obtain $I(\rho)$, which is related to the plasma volume as well as the toroidal magnetic field. In the present 2D transport modeling, the safety factor $q(\rho)$ or $\psi'(\rho)$ and the toroidal current density $j_\zeta(\rho,\chi)$ are fixed in solving Eq. (91) to calculate the derivative of the volume $\mathrm{d}V/\mathrm{d}\rho$. This quantity is used to calculate $j_\zeta(R,Z)$ from $j_\zeta(\rho,\chi)$ before solving Eq. (95)

## 8. Summary and Discussion

The set of equations describing the two-dimensional transport in a whole tokamak plasma has been derived in

MSCS from the multi-fluid equations and Maxwell's equations, where the flux-surface-average has been applied on Eq. (85) and Eq. (91) in order to meet the constraint for the existence of the magnetic surface. The set of the fluid equations consists of the equation for the particle density $n_a$ (29), the parallel momentum $m_a n_a u_{a\|}$ (38), the toroidal momentum $m_a n_a u_{a\zeta}$ (45), the radial momentum $m_a n_a u_a^\rho$ (50), the pressure $p_a$ (56), the parallel total heat flux $Q_{a\|}$ (57), the toroidal total heat flux $Q_{a\zeta}$ (64) and the radial total heat flux $Q_a^\rho$ (69) for each particle species. The set of equations for electromagnetic field includes the poloidal magnetic field $\psi'$ (84), the toroidal magnetic field $I$ (85), the toroidal electric field $E_\zeta$ (91), the poloidal electric field $E_\chi$ (93) and the radial electric field $E_\rho$ (94).

The neoclassical parallel viscosity and heat viscosity have been rewritten in order to be applicable in the open field region outside the LCFS. We have shown that our parallel viscosity is consistent with the Hirshman-type parallel viscosity inside the LCFS and the Braginskii-type one outside the LCFS.

We have shown that our fluid equations are consistent with the neoclassical transport theory by yielding the neoclassical force balance equations from the equations for the parallel momentum and the parallel total heat flux. These equations are expected to provide a better description of the time evolution of the tokamak plasma, especially that of the poloidal and toroidal rotation.

We have emphasized the extension of the neoclassical transport in this article. The turbulent transport induced by the interaction with wave fluctuations will be included similarly as discussed in [7].

We are currently developing a new two-dimensional transport code TASK/T2 based on this model. The procedure for coupling the transport code with the equilibrium code has been described briefly in Sec. 7, where the FCT scheme is employed in order to keep the safety factor unchanged during the calculation of the two-dimensional equilibrium magnetic field. Numerical results with the finite element method and a full-implicit solver will be reported in near future.

# Acknowledgment

We would like to thank Dr. T. Takizuka in Osaka University and Dr. K. Shimizu in Japan Atomic Energy Agency for useful discussions on the peripheral physics and encouragement.

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