On the Ion Distribution Function in Degenerate Electron Plasmas*)

Kohei SUGITA, Hideaki MATSUURA and Yasuyuki NAKAO

Department of Applied Quantum Physics and Nuclear Engineering, Kyushu University, 744 Motooka, Fukuoka 819-0395, Japan

(Received 31 May 2012 / Accepted 7 September 2012)

In highly-compressed plasmas as realized in inertial confinement fusion, the wave nature of electrons becomes noticeable and Pauli's exclusion principle restricts the energy transition of electrons remarkably. Such a state is called "electron degeneracy". In addition, the electron degeneracy may affect the energy distribution and temperature of coexisting ions through Coulombic ion-electron interaction. In order to evaluate these effects, we developed and solved the model equation for the distribution function of ions in degenerate electron plasmas. As a result, it is shown that the ion distribution function maintains a Maxwellian form at a temperature equal to that of degenerate electrons in thermal equilibrium because two effects of electron degeneracy—spectral hardening and Pauli blocking—counteract each other. Furthermore the electron degeneracy slows temperature relaxation between ions and electrons in non-thermal equilibrium.

Keywords: ion distribution function, ion temperature, electron degeneracy, ultra-high density plasma, Coulombic interaction, Pauli blocking

DOI: 10.1585/pfr.8.3404050

1. Introduction

In understanding physics in fusion plasmas, it is important to know precise energy distribution of fuel ions, which is often approximated by a Maxwellian function in thermal equilibrium plasmas, because the rate of fusion reaction strongly depends on the energy distribution of ions. In inertial confinement fusion, the fuel is compressed to ultra-high density as 1000 times the solid density and the mean distance between electrons gets shorter than electron's de Broglie wave length, which is much longer than that of ions. Accordingly the wave nature of electrons becomes noticeable and the quantum effect stands out. Since electron is fermion, the energy transition is restricted by Pauli's exclusion principle and such a situation is called "electron degeneracy". The degree of degeneracy can be evaluated by the degeneracy parameter $\Theta \equiv T_e/E_F$ (E_F is Fermi energy). If $\Theta \leq 1$, the electron degeneracy arises remarkably; the smaller Θ is, the stronger the degeneracy becomes. The consequences of electron degeneracy are as follows:

(a) The electron distribution function gets to follow the Fermi-Dirac statistics and the fraction of high energy components increases than in the case of Maxwellian distribution (spectral hardening).
(b) Scattering collision between an electron and other particle is restricted (Pauli blocking).

In addition, these events may also affect the energy distribution (and temperature) of coexisting ions through Coulombic ion-electron interaction. These effects should be evaluated by calculating the distribution function of ions, but there has been no precedent that the equation for the ion distribution function in which above two effects due to electron degeneracy are perfectly incorporated is solved. Thus we develop the model equation to describe the distribution function of ions coexisting with degenerate electrons and examine the magnitude of these effects by solving it.

2. Kinetic Model Equation

We start from an equation of evolution for the velocity distribution function $f(\boldsymbol{v}, t)$ of ions coexisting with degenerate electrons:

$$\frac{\partial}{\partial t} f(\boldsymbol{v}, t) = \left(\frac{\partial}{\partial t} f\right)_{\text{coll}} - L(\boldsymbol{v}, t) + S(\boldsymbol{v}, t), \tag{1}$$

where $(\partial f/\partial t)_{\text{coll}}$ represents the scattering collision term (in- and out-scattering rates), $L(\boldsymbol{v}, t)$ is the loss rate (if any) due to nuclear reactions, and $S(\boldsymbol{v}, t)$ is the independent source (if any) at arbitrary time t. Hereafter we drop the time variable t.

Usually, small-angle Coulomb scattering term is written in the Fokker-Planck (FP) form, but the FP form is not suitable for describing scattering in individual level; it is difficult to incorporate Pauli blocking, which is effect on the individual scattering, into the final form of the FP term. Therefore we get back to the Boltzmann integral [1] and incorporate Pauli blocking into it. After that, we recover the FP-like form for small-angle Coulomb scattering.

In order to put the equations into more tractable form, we suppose isotropic velocity space and use energy E in-

author's e-mail: sugita@nucl.kyushu-u.ac.jp

*) This article is based on the presentation at the Conference on Inertial Fusion Energy '12.

stead of velocity $\boldsymbol{v}$ as an independent variable and adopt the flux $\Psi(E)$ defined by $\Psi(E) = vf(E)$, where $f(E)$ is the ion energy distribution function. The final form of the equation is

$$\begin{aligned}&\frac{1}{v}\frac{\partial}{\partial t}\Psi(E) + \sum_{j(\neq e)} n_j\sigma_j^{\mathrm{LET}}(E)\Psi(E)\\&= \sum_j \frac{\partial}{\partial E}\left[S_j(E)\Psi(E)\right] + \sum_j \frac{\partial^2}{\partial E^2}\left[D_j(E)\Psi(E)\right]\\&+ \sum_{j(\neq e)} \int n_j\sigma_j^{\mathrm{LET}}(E' \to E)\Psi(E')\mathrm{d}E' + Q(E), \quad (2)\end{aligned}$$

where n_j is the number density of background species $j(j = e, i)$. On the left-hand side, the second term is the removal rate due to large-angle scattering (the symbol "LET" presents "large energy transfer"). We neglected the loss due to absorption. The first and second terms on the right-hand side are the expansions of the small-angle scattering term. The third term represents the large-angle in-scattering rate. The last term is the independent source. The (differential) cross sections in Eq. (2) are the averages over the thermal motion of target j [2].

The functions S_j and D_j are defined during processing the small-angle Coulomb scattering term and written as

$$\begin{aligned}S_j(E) &= \frac{2\pi}{v}\int_0^\infty v_j^2 f_j(v_j)\left[\int_0^\pi v_r \sin\Theta\left\{\int_0^\pi \sin\theta\right.\right.\\&\times\left.\left.\left(\int_0^{2\pi} T\sigma_j(v_r;\theta,\phi)[1-\delta_{je}f_{\mathrm{FD}}(E_e+\delta E_e)]\mathrm{d}\phi\right)\mathrm{d}\theta\right\}\mathrm{d}\Theta\right]\mathrm{d}v_j, (3)\end{aligned}$$

$$\begin{aligned}D_j(E) &= \frac{\pi}{v}\int_0^\infty v_j^2 f_j(v_j)\left[\int_0^\pi v_r \sin\Theta\left\{\int_0^\pi \sin\theta\right.\right.\\&\times\left.\left.\left(\int_0^{2\pi} T^2\sigma_j(v_r;\theta,\phi)\left[1-\delta_{je}f_{\mathrm{FD}}(E_e+\delta E_e)\right]\mathrm{d}\phi\right)\mathrm{d}\theta\right\}\mathrm{d}\Theta\right]\mathrm{d}v_j, (4)\end{aligned}$$

where v_j and f_j are speed and distribution function of background species j, $v_r = |\boldsymbol{v} - \boldsymbol{v}_j|$ ($\boldsymbol{v}_j$ is the velocity of target j), f_{FD} is Fermi-Dirac distribution function as the probability of occupation and T is the amount of energy change in a single scattering. The effects of electron degeneracy are incorporated in these quantities; the electron distribution function f_e ($j = e$) is obtained from the Fermi-Dirac statistics, and the factor $[1 - \delta_{je}f_{\mathrm{FD}}\ (E_e + \delta E_e)]$ represents the probability that the electron gets away from Pauli blocking. Replacing this factor by 1, Eq. (3) and (4) equate to what obtained without including Pauli blocking. The fuel ions move with being shielded by the background electrons that run more rapidly than ions, so we use the cross section including the screening potential [3] as the Coulomb cross section. Simultaneously, in high density plasmas, "strong screening" [4] is also important, so we include its effect by Brysk's approximation for the Debye screening distance [5]. Besides S_j is the same form as the Coulombic stopping power [6] while D_j is the energy dispersion coefficient [7].

3. Results and Discussion

3.1 Steady-state calculation

We suppose steady-state in order to examine the energy distribution of ions in thermal equilibrium. The equation to be solved is hence

$$\begin{aligned}&\sum_{j(\neq e)} n_j\sigma_j^{\mathrm{LET}}(E)\Psi(E)\\&= \sum_j \frac{\partial}{\partial E}\left[S_j(E)\Psi(E)\right] + \sum_j \frac{\partial^2}{\partial E^2}\left[D_j(E)\Psi(E)\right]\\&+ \sum_{j(\neq e)} \int n_j\sigma_j^{\mathrm{LET}}(E' \to E)\Psi(E')\mathrm{d}E' + Q(E). \quad (5)\end{aligned}$$

Supposing steady-state DT plasmas at various electron temperatures T_e and degeneracy parameters Θ, and fixing the distribution function of electrons $f_e(v_e)$, we solved Eq. (5). Figure 1 presents, as an example, the deuteron distribution function when $T_e = 0.5$ keV, $\Theta = 0.1$. The solid curve shows the distribution function calculated by fully considering the effect of electron degeneracy (case1), while the dashed curve is obtained by partially considering the degeneracy effect, that is, the electron distribution function $f_e\ (v_e)$ based on the Fermi-Dirac statistics was used but Pauli blocking was ignored (case2). The dotted curve (almost falling on the solid one) is the result in the case ignoring the electron degeneracy (case3). It can be seen that each ion distribution function forms the Maxwellian distribution. However, the ion temperature gets higher when the degeneracy effect is partially considered.

Table 1 presents the energy flow from electrons to ions in the above examples. In Table 1, "high energy component" presents the electrons with higher than average energy and "lower energy component" does those with lower than average energy. In addition, negative values mean that the energy actually flows from ions to electrons. Since our calculation supposed steady-state, the total flow becomes almost zero.

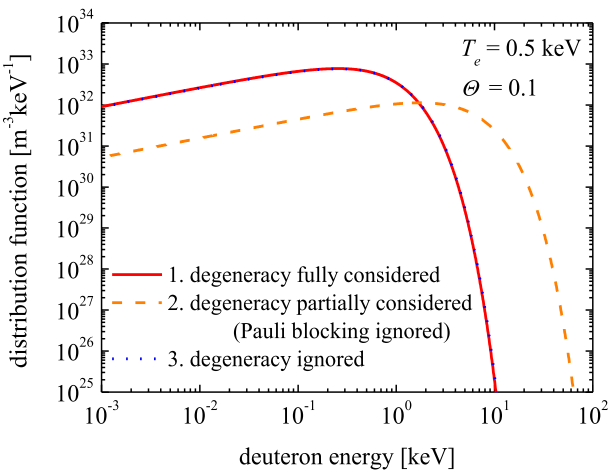

Fig. 1 Deuteron distribution function.

Table 1 Energy flow from electrons to ions [keV/cm^{3}s].

	to high energy component	to low energy component	total (absolute)	total [-] (relative)
fully considered	-1.96×10^{39}	1.96×10^{39}	8.03×10^{35}	0.00041
partially considered	-1.29×10^{40}	1.29×10^{40}	7.33×10^{36}	0.00057
ignored	-9.31×10^{39}	9.31×10^{39}	3.84×10^{36}	0.00041

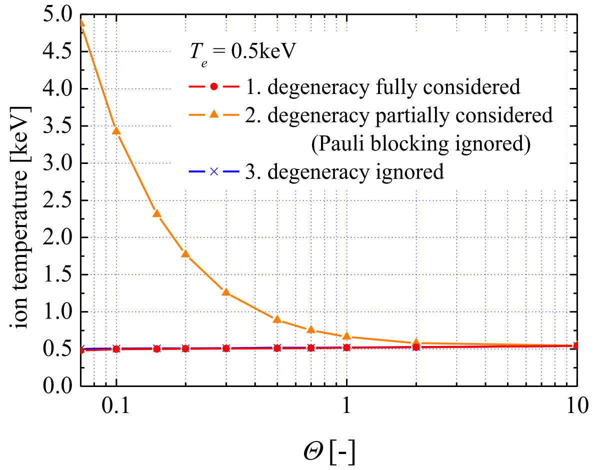

Fig. 2 Ion temperature.

From the calculated ion distribution function, we evaluated the ion temperature:

$$T_\mathrm{i} = \frac{2}{3}\langle E \rangle = \frac{2}{3}\frac{\int_0^\infty E f_\mathrm{i}(E)\mathrm{d}E}{\int_0^\infty f_\mathrm{i}(E)\mathrm{d}E}. \tag{6}$$

Figure 2 presents Θ - dependency of the ion temperature when $T_\mathrm{e} = 0.5$ keV. In the case where the electron degeneracy is ignored, the ion temperature is equal to the electron temperature.

As for how to incorporate the electron degeneracy, we obtain the following results:

(a) The ion temperature gets higher when the degeneracy effect is partially included (ignoring Pauli blocking), and this is significant at low Θ region.

(b) However, the result of (a) disappears when we consider Pauli blocking.

The event like (a) can be explained as follows. When the electron degeneracy arises, the number of electrons in high energy region increases and ions get able to obtain more energy from electrons. As a result, the number of high energy ions increases and the ion distribution function spreads toward high-energy side and the energy flow between ions and electrons gets to balance.

Meanwhile the event like (b) happens for the following reasons. Pauli blocking mainly restricts energy losing of the electrons than energy gaining because of higher probability of electron occupation in the lower energy region. In other words, the energy flow from electrons to ions is more prohibited than that from ions to electrons. In consequence, the energy flow that increased in (a) is completely restricted and the ion temperature does not change from the result without considering the electron degeneracy.

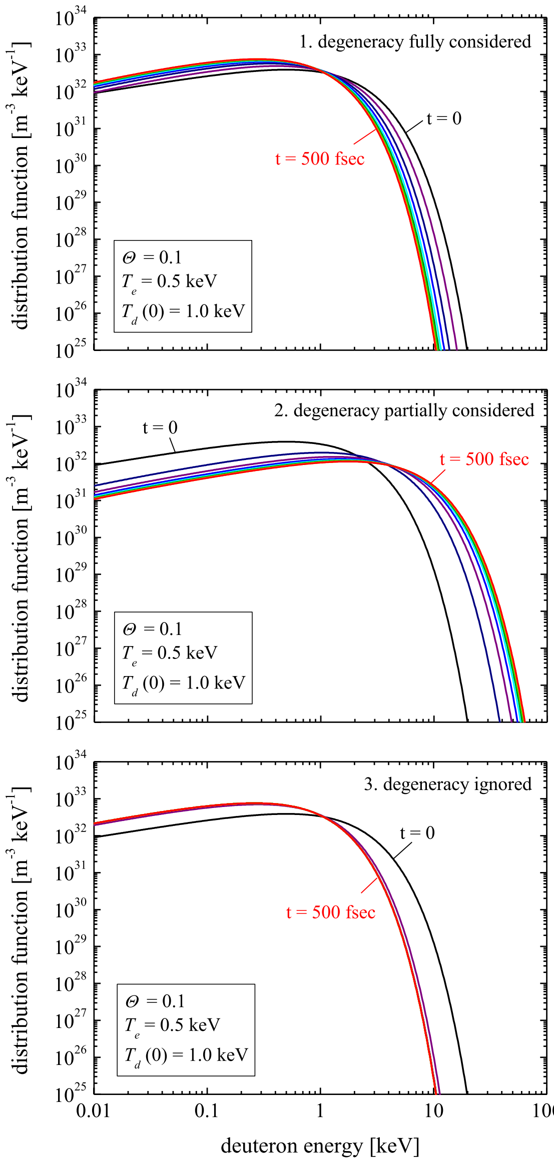

Fig. 3 Deuteron distribution function at every 50 fsec.

3.2 Time-dependent calculation

In order to confirm the results of steady-state calculation, we solved the time-dependent equation, that is, Eq. (2) from a state of non-equilibrium ($T_\mathrm{i}(0) \neq T_\mathrm{e}$) until an equilibrium state with fixing the distribution function of electrons. Figure 3 presents the deuteron distribution function at every 50 fsec, when $T_\mathrm{e} = 0.5$ keV, $\Theta = 0.1$ and the initial ion temperature $T_\mathrm{i}(0)(= T_\mathrm{d} = T_\mathrm{t}) = 1.0$ keV

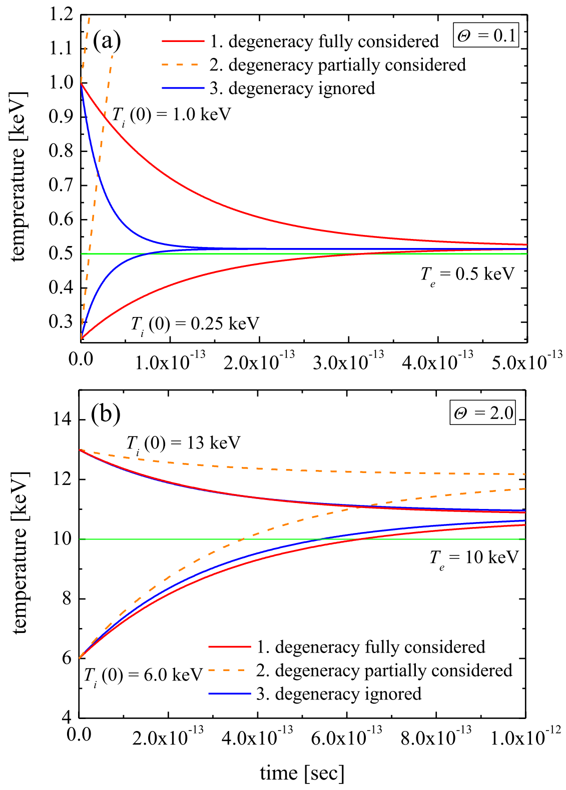

Fig. 4 The time variation of ion temperature: (a) T_e = 0.5 keV and Θ = 0.1, (b) T_e = 10 keV and Θ = 2.0.

in previous three cases (case1–3). In each case, the ion distribution functions approach those obtained by steady-state calculation (Fig. 1) with time, but the time required to reach is extended when degeneracy effects are considered.

For the sake of plain comparison, we evaluated the time variation of ion temperature from the calculated distribution function. Figure 4 presents the results when (a) T_e = 0.5 keV and Θ = 0.1 and (b) T_e = 10 keV, Θ = 2.0. In Fig. 4 (a), the upper curves are calculated from $T_i(0)$ = 1.0 keV as the initial temperature and lower ones are from $T_i(0)$ = 0.25 keV. Figure 4 (b) presents time variation of ion temperature when the electron temperature is higher and Θ is larger than Fig. 4 (a) (the electron density is equivalent). It is confirmed that the ion temperature is equal to the electron temperature when the degeneracy effect is fully considered in thermal equilibrium (however when electron temperature increases, ion temperature gets estimated higher because our calculation does not include dynamical effect [8] of Debye shielding; our model may overestimate the shielding effect for the fast particles, thus preventing them from slowing down). In addition, we can see that the electron degeneracy slows temperature relaxation between ions and electrons. This is because Pauli blocking decreases the amount of energy flow between ions and electrons. This consequence is more noticeable when temperature is lower.

In this paper we focuses on "stationary" ultra-high density plasmas as realized just after laser implosion, on the other hands, the following can be indicated in regard to the fast ignition: In fast ignition highly compressed plasmas, where electron degeneracy can arise, are heated by the short pulse laser in order to increase the ion temperature. In the case that core heating is achieved by fast electrons, as an example, the degenerate electrons in plasmas are initially heated because the electron-electron collision frequency is much larger than the electron-ion collision frequency. Subsequently ions are heated through ion-electron interaction but the electron degeneracy tends to disturb this heating process and the time required to be heated will be lengthened. Such a consequence especially appears in primary phase of heating, during which the electron degeneracy still affects. For the sake of precise evaluation of such an effect, analyses with considering the time dependence of the electron temperature (and also degeneracy parameter Θ), which includes the interaction with fast electrons, for example, are necessary hereafter.

4. Conclusion

By properly incorporating the effect of electron degeneracy, we have found that the ion distribution function maintains Maxwellian form and the ion temperature becomes equal to the electron temperature in thermal equilibrium even if electrons are in degenerate state. This is because two effects of electron degeneracy—spectral hardening and Pauli blocking—counteract each other. In addition, the electron degeneracy slows temperature relaxation between ions and electrons, so it is necessary to include the degeneracy effect when analyzing the ion heating process.

[1] See, for example, F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, New York, 1965).
[2] Y. Nakao, K. Tsukida and V.T. Voronchev, Phys. Rev. D **84**, 063016 (2011).
[3] T. Kammash, *Fusion Reactor Physics* (Ann Arbor Science Publishers, Ann Arbor, MI, 1975) p.86.
[4] E.E. Salpeter, Aust. J. Phys. **7**, 373 (1954).
[5] H. Brysk, P.M. Campbell and P. Hammerling, Plasma Phys. **17**, 473 (1974).
[6] D.V. Sivukhin, *Review of Plasma Physics*, Vol. 4 (Consultants Bureau, New York, 1966).
[7] S.T. Butler and M.J. Buckingham, Phys. Rev. **126**, 1 (1962).
[8] See, for example, D.R. Nicholson, *Introduction to Plasma Theory* (John Wiley & Sons, Inc., New York, 1983).