# Inclusion of Pressure and Flow in the KITES MHD Equilibrium Code*)

Daniel RABURN and Atsushi FUKUYAMA

*Department of Nuclear Engineering, Kyoto University, Kyoto 606-8501, Japan*

(Received 10 December 2012 / Accepted 5 March 2013)

One of the simplest self-consistent models of a plasma is single-fluid magnetohydrodynamic (MHD) equilibrium with no bulk fluid flow under axisymmetry. However, both fluid flow and non-axisymmetric effects can significantly impact plasma equilibrium and confinement properties: in particular, fluid flow can produce profile pedestals, and non-axisymmetric effects can produce islands and stochastic regions. There exist a number of computational codes which are capable of calculating equilibria with arbitrary flow or with non-axisymmetric effects. Previously, a concept for a code to calculate MHD equilibria with flow in non-axisymmetric systems was presented, called the KITES (Kyoto ITerative Equilibrium Solver) code [D. Raburn and A. Fukuyama, Plasma Fusion Res. **7**, 240318 (2012)]. Since then, many of the computational modules for the KITES code have been completed, and the work-in-progress KITES code has been used to calculate non-axisymmetric force-free equilibria [D. Raburn and A. Fukuyama, Proceedings of the 9th EPS Conference on Plasma Physics, Stockholm, Sweden (2012)]. Additional computational modules are required to allow the KITES code to calculate equilibria with pressure and flow. Here, the authors report on the approaches used in developing these modules and provide a sample calculation with pressure.



Keywords: magnetohydrodynamics, equilibrium, calculation, flow, island

DOI: 10.1585/pfr.8.2402030

## 1. Introduction

The concept for the KITES code was presented in Ref. [1,2]. Four major hurdles were identified, which can be expressed as: 1) Calculating a field-line label and the topology of the plasma; 2) Solving the inhomogeneous magnetic differential equation (IMDE) ($\boldsymbol{B} \cdot \boldsymbol{\nabla} f = g \neq 0$); 3) Handling the sonic discontinuity; and, 4) Working around the hyperbolicity of single-fluid MHD equilibria with general flows under adiabatic closure. The first two hurdles have essentially be cleared, but, since publication of the concept, a complication has been identified for each: A) Calculating a modified field-line label near separatrices to have continuous derivatives of the free profiles; and, B) Accurately solving the IMDE in stochastic regions.

Complication A) is addressed in Sec. 2.

For complication B), if possible, IMDEs in stochastic regions should be recast in to homogeneous form. In the stationary case, there is no IMDE in stochastic regions, so this is not a problem. The work-in-progress KITES code is capable of calculating stationary MHD equilibria with pressure, and a sample calculation is presented in Sec. 3.

Turning out attention to flowing equilibria, complication B) has necessitated some change to the KITES concept. In the case with purely parallel flow and adiabatic closure, we have found that it is indeed possible to avoid having to solve an IMDE in stochastic regions, so long as all of the current inside the plasma is carried by the plasma itself. A suitable formulation for such equilibria is presented in Sec. 4.

## 2. Island Modified Field-Line Label

KITES makes use of a field-line label, $\psi$. On each field line which constitutes a good flux surface, we take $\psi$ to be:

$$\psi(line) = \sqrt{V_{\text{good}}(line)/V_{\text{good LCFS}}} \tag{1}$$

where $V_{\text{good}}$ is the amount of "good" volume inside a good flux surface (total volume minus volume inside islands and stochastic regions) and $V_{\text{good LCFS}}$ is the good volume inside the last closed flux surface. Inside islands and stochastic regions, we take $\psi$ to be the value on the separatrix. $\psi$ is calculated by following several trial lines for many toroidal circuits and applying a heuristic to the puncture points on each $\phi = cnst$ plane to determine if they form a good flux surface; this information is then interpolated to the fixed $(R, Z, \phi)$ grid.

Near a magnetic island, $\boldsymbol{\nabla}\psi$ may be discontinuous or infinite. Under a simple model of an island, going out from the magnetic axis towards an o-point, $\psi \sim \psi_{\text{sepx}} - c\delta^{1/2}$, where $\psi_{\text{sepx}}$ is the value of $\psi$ on the separatrix, $c$ is some constant, and $\delta$ is the distance from the separatrix. Because the finite-differencing in KITES is not designed to handle discontinuities, we wish to construct a modified $\psi$ near the

*author's e-mail: draburn@p-grp.nucleng,kyoto-u.ac.jp*

*) This article is based on the presentation at the 22nd International Toki Conference (ITC22).

island such that it is continuous in physical space. Based on the size of the island, we pick some $\psi_0$ and $\psi_1$ with $\psi_0 < \psi_{\mathrm{sepx}} < \psi_1$ and calculate an "island modified" (isl mod) $\psi$ in that region.

$$\psi_{\mathrm{isl\ mod}} = \psi_{\mathrm{sepx}} - \left(\psi_{\mathrm{sepx}} - \psi_0\right) F\left(\frac{\psi_{\mathrm{sepx}} - \psi}{\psi_{\mathrm{sepx}} - \psi_0}\right) \quad (2)$$

$$\psi_{\mathrm{isl\ mod}} = \psi_{\mathrm{sepx}} + \left(\psi_1 - \psi_{\mathrm{sepx}}\right) F\left(\frac{\psi - \psi_{\mathrm{sepx}}}{\psi_1 - \psi_{\mathrm{sepx}}}\right) \quad (3)$$

where Eq. (2) is for $\psi_0 < \psi < \psi_{\mathrm{sepx}}$, Eq. (3) is for $\psi_{\mathrm{sepx}} < \psi < \psi_1$, and $F(X)$ depends on the desired degree of continuity:

$$F(X) \equiv \begin{cases} 3X^3 - 2X^4 & \text{for } C^1 \\ 15X^5 - 24X^6 + 10X^7 & \text{for } C^2 \end{cases} \quad (4)$$

# 3. Calculation of Stationary Equilibria with Pressure

## 3.1 Formulation and numerical algorithm

First, we break up $\boldsymbol{B}$ and $\boldsymbol{j}$ into "applied" and "plasma" parts ($\boldsymbol{B} = \boldsymbol{B}_{\mathrm{appl}} + \boldsymbol{B}_{\mathrm{plas}}$, etc) and break up $\boldsymbol{j}_{\mathrm{plas}}$ in to parallel and perpendicular parts: $\boldsymbol{j}_{\mathrm{plas}} = \lambda \boldsymbol{B} + \boldsymbol{j}_{\mathrm{plas}\perp}$. The governing equations can be written:

$$\boldsymbol{B} \cdot \nabla p = 0 \quad (5)$$

$$\boldsymbol{j}_{\mathrm{plas}\perp} = \left(\boldsymbol{B} \times \nabla p\right) / B^2 \quad (6)$$

$$\boldsymbol{B} \cdot \nabla \lambda = -\nabla \cdot \boldsymbol{j}_{\mathrm{plas}\perp} \quad (7)$$

$$\boldsymbol{B}_{\mathrm{plas}} = \mu_0 [\nabla \times]^{-1} \boldsymbol{j}_{\mathrm{plas}} \quad (8)$$

with the boundary condition $\boldsymbol{B}_{\mathrm{plas}} \to 0$ at $\infty$. We use $p = f_p(\psi_{\mathrm{isl\ mod\ C2}})$ and $\lambda = f_\lambda(\psi) + \lambda_{\mathrm{var}}$ for some $f_p$ and $f_\lambda$, where $\lambda_{\mathrm{var}}$ must satisfy:

$$\boldsymbol{B} \cdot \nabla \lambda_{\mathrm{var}} = -\nabla \cdot \boldsymbol{j}_{\mathrm{plas}\perp} \quad (9)$$

$$\langle \lambda_{\mathrm{var}} \rangle = 0 \quad (10)$$

where $\langle ... \rangle$ indicates the average [ $(minval + maxval)/2$ ] over the field-line.

The homogeneous magnetic differential equation for $p$ [Eq. (5)] is solved everywhere by calculating $\psi_{\mathrm{isl\ mod\ C2}}$ as in Sec. 2 and applying the given $f_p$. (Note that, because $\boldsymbol{B} \cdot \nabla \psi_{\mathrm{isl\ mod\ C2}} = 0$ by construction, $\boldsymbol{B} \cdot \nabla p = 0$ is satisfied. For more information, see Ref. [1].)

Turning our attention to $\lambda_{\mathrm{var}}$, observe that, inside islands and stochastic regions: $\psi = cnst \implies p = cnst \implies \boldsymbol{j}_{\mathrm{plas}\perp} = 0 \implies \boldsymbol{B} \cdot \nabla \lambda_{\mathrm{var}} = 0 \implies \lambda_{\mathrm{var}} = 0$. Hence, there is no no need to solve the inhomogeneous magnetic differential equation for $\lambda_{\mathrm{var}}$ [Eq. (9)] inside islands and stochastic regions. On good field lines, we calculate $\lambda_{\mathrm{var}}$ by integrating along the field line:

$$\lambda_{\mathrm{var}}(l + dl) = \lambda_{\mathrm{var}}(l) - dl\left(\nabla \cdot \boldsymbol{j}_{\mathrm{plas}\perp}\right) / B + cnst. \quad (11)$$

where $l$ is the length along the field line. When solving Eq. (11), we start on the outboard mid-plane at $\phi = 0$ and temporarily set $\lambda_{\mathrm{var}}(0) = 0$; after many circuits, we calculate the average of the temporary $\lambda_{\mathrm{var}}$ then add a constant to make the average zero. (Note that this method of solving the IMDE is simpler than the method proposed in Ref. [1].)

## 3.2 Sample calculation

We now present a sample calculation. The applied vector potential ($\nabla \times \boldsymbol{A}_{\mathrm{appl}} = \boldsymbol{B}_{\mathrm{appl}}$) is:

$$\begin{aligned} R\boldsymbol{A}_{\mathrm{appl}} &= 10^{-3} F(r) \left[1 + 5F(r)\cos(3\theta)\right] \\ &\quad Z\,\hat{\boldsymbol{R}} + \tfrac{1}{5} r^2 \cos(\phi) \left[\cos(\theta)\,\hat{\boldsymbol{Z}} - \sin(\theta)\,\hat{\boldsymbol{R}}\right] \end{aligned} \quad (12)$$

where $r \equiv \sqrt{(R-4)^2 + Z^2}$, $\theta \equiv \mathrm{atan2}(Z, R-4)$, and $F(r) \equiv [r^2(1-r^2)^2]^2$. This field has a 3/0 island and a radial squeeze/expansion with varying $\phi$, as shown in Figs. 2 and 3. Note that $\boldsymbol{j}_{\mathrm{appl}} \neq 0$. The free profiles are: $f_\lambda(\psi) = \lambda_0(1-\psi^4)^2$ and $f_p(\psi_{\mathrm{islmod\ C2}}) = (1 - \psi^4_{\mathrm{islmod\ C2}})^8$ with $\lambda_0 = 5 \times 10^{-5}$ and $p_0 = 8 \times 10^{-11}$. The vacuum vessel cross-section is square: $R = 3 \sim 5$, $Z = -1 \sim 1$. The $(R, Z, \phi)$ cylindrical mesh is over the range $(2.9 \sim 5.1,\ -1.1 \sim 1.1,\ 0 \sim 2\pi)$ with $101 \times 101 \times 50$ grid points. ($101 \times 101$ in the $RZ$ plane gives a mesh spacing on the order of typical ion gyroradius. 50 grid points in $\phi$ was chosen anticipating that, when taking the Fourier transform of a quantity, only a small fraction of the power would be in the upper half of the spectrum; this was verified a posteriori, with the higher modes of $|B_{\mathrm{plas}}|$ contributing to only $2.5 \times 10^{-5}$ of the total.)

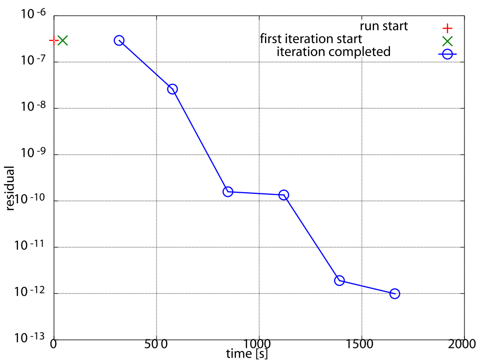


Fig. 1 Plot of $L_2$ residual vs time for sample calculation.

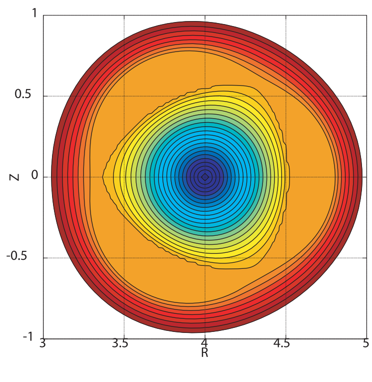


Fig. 2 Contour plot of $\psi$ for the applied field on the plane $\phi = 0$.

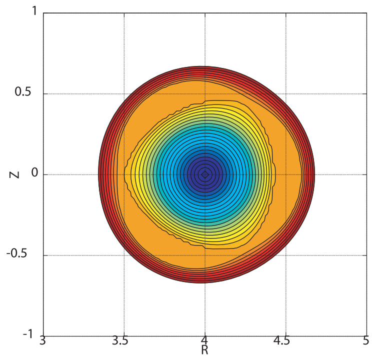


Fig. 3 Contour plot of $\psi$ for the applied field on the plane $\phi = \pi$.

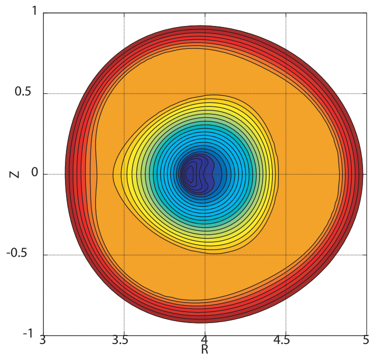


Fig. 4 Contour plot of equilibrium $\psi$ on the plane $\phi = 0$.

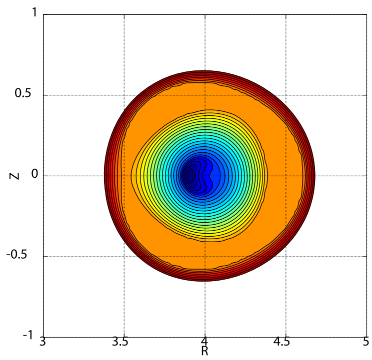


Fig. 5 Contour plot of equilibrium $\psi$ on the plane $\phi = \pi$.

The calculation was run on a single core of an Intel Core i5-2400 3.1GHz and converged in 28 minutes with 6 iterations with no relaxation, from a residual of $3 \times 10^{-7}$ to $1 \times 10^{-12}$, as shown in Fig. 1. The equilibrium $\psi$ (Fig. 6), $\lambda$ (Fig. 7), and $p$ (Fig. 8) are shown in the remaining figures. From Figs. 4 and 5, observe that the magnetic axis shifts inward and the shape of the magnetic surfaces near the axis are deformed, but the impact of toroidal variation is unchanged. From Fig. 7, observe that the pressure (via $\lambda_{\mathrm{var}}$) makes a large contribution to $\lambda$.

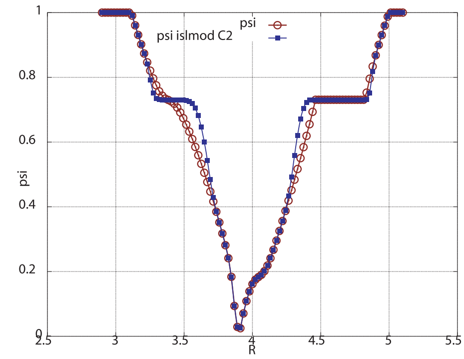


Fig. 6 Plot of equilibrium $\psi$ and $\psi_{\mathrm{islmod\ C2}}$ along the mid-plane at $\phi = 0$.

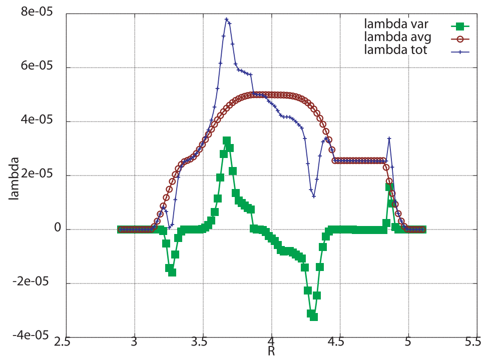


Fig. 7 Plot of equilibrium $\lambda_{\mathrm{avg}} \equiv f_\lambda$, $\lambda_{\mathrm{var}}$, and $\lambda_{\mathrm{tot}} \equiv \lambda = f_\lambda + \lambda_{\mathrm{var}}$ along the mid-plane at $\phi = 0$.

## 4. Formulation of Equilibria with Purely Parallel Flow

Define:

$$M_{\mathrm{S}} \equiv v/C_{\mathrm{S}} = \sqrt{m_{\mathrm{i}} v^2/(\gamma T)} \tag{13}$$

$$M_{\mathrm{A}} \equiv v/V_{\mathrm{A}} = \sqrt{\mu_0 m_i n v^2/B^2} \tag{14}$$

$$\tau \equiv \tfrac{\gamma}{\gamma-1} p/n^\gamma \tag{15}$$

$$\omega \equiv v_\parallel n/B \tag{16}$$

$$\kappa \equiv \tfrac{1}{2} m_{\mathrm{i}} \omega^2 \tag{17}$$

$$E \equiv \tfrac{\gamma}{\gamma-1} T + K = \gamma T \left(\tfrac{1}{\gamma-1} + \tfrac{1}{2} M_{\mathrm{S}}^2\right) \tag{18}$$

where $T \equiv p/n$ and $K \equiv \frac{1}{2} m_{\mathrm{i}} v^2$. As in the stationary case, we use $\lambda \equiv j_\parallel/B$ and let $\psi$ be a field-line label.

The equations for conservation of particles, adiabatic equation of state, and parallel force-balance become:

$$\boldsymbol{B} \cdot \boldsymbol{\nabla} \omega = 0 \tag{19}$$

$$T \omega \boldsymbol{B} \cdot \boldsymbol{\nabla} \log(\tau) = 0 \tag{20}$$

$$\boldsymbol{B} \cdot \left[ n \boldsymbol{\nabla} E - (\gamma - 1)^{-1} p \boldsymbol{\nabla} \log(\tau) \right] = 0 \tag{21}$$

Requiring $\boldsymbol{B} \neq 0$ inside the plasma, it is clear that $\omega = f_\omega(\psi)$ and $\kappa = f_\kappa(\psi) = \frac{1}{2} m_i f_\omega^2(\psi)$ for some function $f_\omega$.

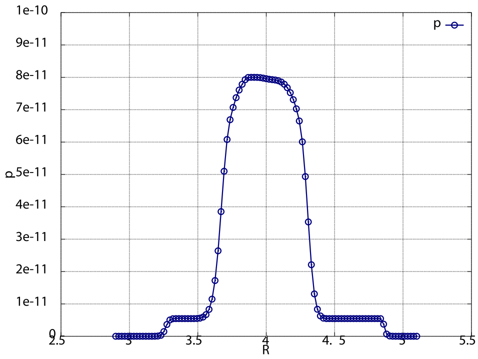


Fig. 8 Plot of equilibrium $p$ along the mid-plane at $\phi = 0$.

Further requiring $T \neq 0$ and $\omega \neq 0$, it is clear that $\tau = f_\tau(\psi)$ and $E = f_E(\psi)$ for some functions $f_\tau$ and and $f_E$. (More accurately, because derivatives of all of these functions will enter in to the equations, they should be taken as explicit functions of $\psi_{islmod}$.) Now, consider the case $\omega = 0$: this is the stationary case, in which $p = f_p(\psi)$; $f_E$ and $f_\tau$ are not independently determined, and we are free to take both to be functions of $\psi$. We will not attempt to handle the case $\omega \neq 0, T = 0$.

The definition of $E$ yields the Bernoulli equation, which relates the density, magnetic topology, and magnetic field strength:

$$E = \tau n^{\gamma-1} + \kappa B^2 n^{-2} \tag{22}$$

As in the axisymmetric case, for a given $\psi$ and $B$, Eq. (22) can have zero, one, or two real solutions for $n$, assuming $\tau \neq 0$ and $\kappa B^2 \neq 0$ [3]. For KITES, we define the criteria in terms of a critical energy $E_{crit}(\psi, B)$, which can be conveniently written:

$$\left[\frac{E_{crit}(\psi, B)}{\gamma+1}\right]^{\gamma+1} = \left[\frac{\tau(\psi)}{2}\right]^2 \left[\frac{\kappa(\psi)B^2}{\gamma-1}\right]^{\gamma-1} \tag{23}$$

There are no real solutions for $n$ when $E < E_{crit}(\psi, B)$.

Turning to the perpendicular plasma current density, we write derivatives of $\boldsymbol{B}$ in terms of $\mu_0 \boldsymbol{j}$ and $\nabla B$ and, using Eq. (22), derivatives of $n$ in terms of $\nabla\psi$ and $\nabla B$:

$$\boldsymbol{j}_{plas\perp} = \frac{M_A^2}{1-M_A^2}\boldsymbol{j}_{appl\perp} + \boldsymbol{B} \times \left(F_\psi \nabla\psi + F_B \nabla B\right) \tag{24}$$

$$F_\psi \equiv \frac{p}{(1-M_A^2)B^2}\left[F_{\psi 1} + F_{\psi 2} + F_{\psi 3}\right] \tag{25}$$

$$F_{\psi 1} \equiv \frac{1}{\gamma-1}\left(\gamma \frac{1}{E}\frac{dE}{d\psi} - \frac{1}{\tau}\frac{d\tau}{d\psi}\right) \tag{26}$$

$$F_{\psi 2} \equiv \left(\frac{\gamma}{\gamma-1}\right)\left(\frac{M_S^2}{1-M_S^2}\right)\left(\frac{1}{E}\frac{dE}{d\psi} - \frac{1}{\tau}\frac{d\tau}{d\psi}\right) \tag{27}$$

$$F_{\psi 3} \equiv \left(\frac{\gamma}{2}\right)\left(\frac{M_S^4}{1-M_S^2}\right)\left(\frac{1}{E}\frac{dE}{d\psi} - \frac{1}{\kappa}\frac{d\kappa}{d\psi}\right) \tag{28}$$

$$F_B \equiv \frac{p}{(1-M_A^2)B^2}\left[\frac{-\gamma M_S^4}{1-M_S^2}\frac{1}{B}\right] \tag{29}$$

The $M_A^2 \boldsymbol{j}_{appl\perp}/(1-M_A^2)$ term in Eq. (24) is due to any applied current, and the term will typically be zero.

Note that, unlike in the stationary case, $\nabla \cdot \boldsymbol{j}_{plas\perp}$ typically does not vanish inside stochastic regions. Taking $\nabla\psi = 0$ and dropping the $\boldsymbol{j}_{appl}$ term:

$$\nabla \cdot \boldsymbol{j}_{plas\perp} = \mu_0 \lambda F_B \boldsymbol{B} \cdot \nabla B \tag{30}$$

where we have used the fact that $F_B = F_B(\psi, B, n(\psi, B))$ via Eq. (22). Assuming none of the terms are zero, we cannot accurately solve the IMDE for $\lambda$ [Eq. (7) or (9)] in stochastic regions. Additionally, solving the IMDE for $\lambda$ inside magnetic islands would require accurate determination of the flux surfaces inside islands, which is not presently done in KITES.

However, using Eq. (22), integration rules, and variable substitution, Eq. (7) can be cast in to a homogeneous form by defining $G_B$:

$$G_B(\psi, B, n) \equiv \int -\mu_0 F_B(\psi, B, n)dB \Big|_{\psi=cnst} \tag{31}$$

$$\boldsymbol{B} \cdot \nabla\left(\log\lambda - G_B\right) = 0 \tag{32}$$

where $n = n(\psi, B)$ per Eq. (22). The integral can be evaluated by changing the variable of integration from $B$ to $n$, followed by much algebra:

$$G_B = -\log(1-M_A^2) + cnst \tag{33}$$

Hence, inside stochastic regions, $\lambda$ must have the form:

$$\lambda = \lambda_0 / \left(1 - M_A^2\right) \tag{34}$$

For each island and stochastic region, $\lambda_0$ should be taken to make $\lambda$ as continuous as possible across the separatrices, such as by minimizing $\langle[\lambda_0/(1-M_A^2) - f_\lambda - \lambda_{var}]^2\rangle$ on the separatrices.

## 5. Summary

The first two hurdles in the KITES concept have been cleared, though, in the case with cross-field flows, there may be some complications for accurately solving the inhomogeneous magnetic differential equation in stochastic regions. However, we have shown that this is not a problem for single-fluid MHD equilibria with purely parallel flows. The work-in-progress KITES code has been used to calculate stationary single-fluid MHD equilibria.

This work was supported by Grand-in-Aid for Scientific Research (S) (20226017) from JSPS, Japan.

[1] D. Raburn and A. Fukuyama, Plasma Fusion Res. **7**, 240318 (2012).
[2] D. Raburn and A. Fukuyama, Proceedings of the 9th EPS Conference on Plasma Physics, Stockholm, Sweden (2012).
[3] D. Raburn and A. Fukuyama, Plasma Fusion Res. **6**, 243044 (2011).