Reduction of Nonlinear Effects in Magnetohydrodynamic and Hall Magnetohydrodynamic Turbulence

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1. Introduction

Magnetohydrodynamics (MHD) turbulence is relevant to a wide range of research topics, including astrophysics and fusion plasma studies. In recent years, extensions of MHD equations to two-fluid effects have attracted much attention [1].

A simple fluid model incorporating a two-fluid effect is the Hall MHD (HMHD) system. The MHD and HMHD systems differ significantly in their energy dissipation tendencies, structure formation processes, and profiles of the generated coherent structures [2]. It appears that the Hall term maintains large-scale spatial structures in fully developed turbulence, which is important for plasma confinement.

We also compared the wavelet-scale spectra of the energy exchange generated by the induction term in a snapshot dataset. Local energy transfer dominated the exchange and transfer to small scales was observed in both cases [3]. By analyzing several serial snapshot datasets, we also found that the Hall term induces nonlinear energy transfer over large scales, a phenomenon that is intrinsic to HMHD dynamics [4]. In particular, nonlocal interactions between the modes of a magnetic field are crucial to back scattering phenomena.

Despite these advances, how the nonlocal interaction of the Hall term affects the energy transfer processes in HMHD and MHD systems is yet to be clarified. In the present study, we investigate the temporal dynamics of the energy transfer processes in HMHD and compare them with those of MHD over longer time intervals than in our previous studies.

2. Basic Equations and Numerical Simulation

To exclude forcing influences, we analyze a freely decaying, homogeneous, isotropic turbulence in an incompressible HMHD system. The incompressible HMHD equations are given by

\[ \frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla P + j \times b + \nu \nabla^2 u, \]

\[ \frac{\partial b}{\partial t} = \nabla \times ((u - \epsilon j) \times b) + \eta \nabla^2 b, \]

where the bulk velocity field \( u \) satisfies \( \nabla \cdot u = 0 \), \( b \) is the magnetic field, \( j := \nabla \times b \) is the current density field, \( P \) is the total pressure, \( \nu \) is the kinematic viscosity, \( \eta \) is the resistivity, and \( \epsilon \) is parameter specifying the relative strength of the Hall term. The numerical schemes are detailed in [2].

The parameters are set to \( \nu = \eta = 1 \times 10^{-3} \) and \( \epsilon = 0.05 \). In a comparison simulation, we set \( \epsilon = 0 \). Calculations were performed on \( N^3 = 512^3 \) grid points. We adopted the pseudospectral method with a 2/3 dealiasing rule (\( k_{\text{max}} = 170 \)). To elucidate the effects of the Hall term on the solutions, both simulations were conducted under the same initial conditions.

3. Analysis Method

Since a freely decaying turbulence was imposed, the amplitudes and characteristic scales of turbulent velocity and magnetic fields gradually altered over time. To quan-
where $k$ is the characteristic oscillation wavelength in the wavenumber space: 

$$f$$

ponent is the sum of the Fourier modes on a spherical shell 

$$f$$


tif) δ(t). As shown in [4], the amplitudes of the energy spectra and the energy transfer spectra are of comparable order among the different time snapshots.

To evaluate the shell-averaged energy budget imposed by mode interactions, each field was decomposed into band-pass-filtered components as

$$f(\vec{x}, t) = \sum f_j(\vec{x}, t),$$

where $f$ incorporates the velocity and magnetic fields and $j$ is the scale index of the filtered field. Each filtered component is the sum of the Fourier modes on a spherical shell in the wavenumber space:

$$f_j(\vec{x}, t) := \sum_{\vec{k} \in \Delta_j} f(\vec{k}, t) \exp(2\pi i \vec{k} \cdot \vec{x}/L),$$

where $k_j(t) := \sqrt{\epsilon_{\text{fil}}(t)/\eta^2}$ is the characteristic wavenumber of the dissipation range and $L$ is the system size. The characteristic oscillation wave length $\lambda_j$ and the envelope scale of $f_j$ (also known as the window width of the band pass filter) $\Delta_j$ obey the following scaling relations

$$\lambda_j \propto \sqrt{\Delta_j}, \quad \Delta_j \propto \sqrt{\Delta_j}.$$

Alternatively, geometric series shell partitioning in Fourier space is achieved by convolution with a mollifier:

$$f_j(\vec{x}, t) = \left(\frac{k_j}{\sqrt{\Delta_j}}\right)^3 \int \psi\left(\frac{k_j |\vec{y}|}{\sqrt{\Delta_j}}\right) f(\vec{x} - \vec{y}, t) d^3 \vec{y},$$

where the mollifier $\psi$ is given by

$$\psi(r) = \frac{4\pi}{r^2} \left[r \cos r - \sin r - \sqrt{2} r \cos \sqrt{2} r + \sin \sqrt{2} r\right]$$

(see Fig. 1). We remark that this decomposition is analogous to “dyadic wavelet” analysis [5].

In the present study, we analyzed the energy budget in the band-pass-filtered representation. The energy budget of MHD and HMHD turbulence was analyzed as described in [3], except that the wavelet scale decomposition was replaced with the $\sqrt{2}$-adic scale decomposition given by Eq. (3).

$$\frac{d}{dt} E_j^{(a)} = \sum_{k,m} \langle u_j | u_m u_k \rangle_{\text{adv}} + \sum_{k,m} \langle u_j | b_m | b_k \rangle_{\text{Lor}} + D_j,$$

$$\frac{d}{dt} E_k^{(b)} = \sum_{j,m} \langle b_j | b_m u_j \rangle_{\text{Ind}} + \sum_{j,m} \langle b_j | b_m | b_j \rangle_{\text{Hall}} + R_k,$$

where

$$E_j^{(a)} := \frac{1}{2} \int \langle u_j | u_j \rangle d^3 \vec{x}, \quad D_j := -\nu \int \langle \nabla \times u_j \rangle ^2 d^3 \vec{x},$$

$$E_k^{(b)} := \frac{1}{2} \int \langle b_k | b_k \rangle d^3 \vec{x}, \quad R_k := -\eta \int \langle \nabla \times b_k \rangle ^2 d^3 \vec{x},$$

$$\langle u_j | u_m u_k \rangle_{\text{adv}} := -\langle (u_m \cdot \nabla) u_j \rangle \langle u_j \rangle d^3 \vec{x},$$

$$\langle u_j | b_m | b_k \rangle_{\text{Lor}} := \langle j \times b_m \rangle \langle u_j \rangle d^3 \vec{x},$$

$$\langle b_j | b_m u_j \rangle_{\text{Ind}} := \langle \nabla \times (u_j b_m) \rangle \langle b_j \rangle d^3 \vec{x}, \quad \langle b_j | b_m | b_j \rangle_{\text{Hall}} := -\epsilon \langle \nabla \times (j_k \times b_m) \rangle \langle b_j \rangle d^3 \vec{x}.$$
the Hall term effect. The characteristics of these snapshot datasets probably reflect self-similar features of spontaneously generated MHD and HMHD turbulence. Throughout the analyzed time period, each functional form of the normalized Fourier spectra collapsed into a single form, indicating that the turbulent MHD and HMHD fields both exist in self-similar states (see Fig. 3).

Figures 4 and 5 plot the quadratic terms of the shell-averaged energy transfers. As expected, normalization renders the amplitudes of the energy transfer functions comparable between different time snapshots in both HMHD and MHD. Thus, the key physical parameters are the dissipation coefficient and the energy dissipation rate. In other words, the principal dynamic features are dominated by those of the dissipation range. Here, the physical quantities are normalized by the resistivity coefficient and dissipation rate of magnetic energy. However, normalization by the viscosity coefficient and dissipation rate of kinetic energy yields qualitatively similar results.

Moreover, the peaks of the transfer functions generated by the Lorentz force and magnetic induction are closely spaced in the HMHD and MHD solutions. Peak values of transfer by the Lorentz force are close to each other in both systems. The same tendency is seen for the magnetic induction effect.

In both systems, the energy is overall transferred to smaller scales, while weak inverse transfer to larger scales is observed in the fluid advection effect. All the analyzed snapshots show inverse energy transfer induced by the Hall term.

In the HMHD simulations (see Fig. 4), the energy transfer spectra of the Lorentz force and magnetic induction terms show similar form for $k \gtrsim 0.1k_d(t)$. This suggests that over dissipation range scales, the statistical features of the mode interaction between the normalized magnetic and velocity fields rapidly converge to the equilibrium state.

On the contrary, although the normalized spectra are of comparable amplitude, the energy transfers induced by fluid advection and the Hall term, which rearrange the kinetic and magnetic energy distribution, vary more slowly than those induced by their mutual interactions.
An interesting feature of these systems is that although the energy transfers induced by the Lorentz force effect and the Hall term both involve the $j\times b$ field in their integrands, they converge to the stationary state at very different time scales.

In the MHD medium, energy transfer induced by the Lorentz force and magnetic induction is rapid and more effective than in the HMHD medium. The amplitude of the energy transfer due to fluid advection converges slightly more slowly than that of the other transfers. However, significant reduction in energy transfer, highly obvious in the HMHD medium, is not observed in the MHD medium.

5. Discussion

We compared the time series of the Fourier band-pass filtered spectra of the energies and the quadratic terms in the energy transfers in simulated MHD and HMHD systems. Although the normalized energy spectra converge...
to particular profiles, the amplitude of the energy transfers changes in time. The transfers related to mode interactions between the magnetic and velocity fields converges relatively quickly over dissipation range scales, while the amplitudes of nonlinear terms (fluid advection and the Hall term effect) relax relatively slowly. The relaxation times of both terms are prolonged in HMHD medium, suggesting that slow dynamical modes exist in the decay process.

This slow relaxation following convergence of the energy spectra seems to be related to the so-called “depression of nonlinearity” in fully developed turbulence. This phenomenon has aroused great interest among turbulence researchers and has been investigated in both neutral fluids (e.g. Refs.[7] and [8]) and magnetohydrodynamic (MHD) fluids [9]. The authors of these studies compared turbulent fields with phase randomized fields. In the present study, we found selective depression of triad mode interactions in HMHD turbulence by directly comparing the normalized triad interactions.

Since the amplitudes of the magnetic and velocity fields are normalized, the datasets collected at different times differ principally in the topological features of their vector fields. At present, we are investigating the physical reasons for this novel selective reduction phenomenon.

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