Relativistic Guiding-Center Equations Including Slow Equilibrium Changes in Magnetic Coordinates

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Guiding-center equations for relativistic particles are presented in axisymmetric toroidal geometry using Boozer coordinates. Effects of slow equilibrium changes are included for describing electron acceleration due to the induction field, which is a fundamental process of runaway electron generation during disruptions. For a consistent treatment of the runaway orbit in finite-pressure plasmas, the equations are given in both canonical and noncanonical forms by retaining the radial covariant component of the equilibrium magnetic field. For this purpose, the Lagrangian formulation by White and Zakharov [R.B. White and L.E. Zakharov, Phys. Plasmas **10**, 573 (2003)] is applied to axisymmetric equilibria with slowly varying magnetic-flux functions.

Keywords: guiding-center equation, Boozer coordinate, Hamiltonian and Lagrangian mechanics, runaway electron

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1. Introduction

Guiding-center calculations of particle motion [1] are a powerful way to study energetic particle confinement in toroidal devices. While a majority of work has focused on nonrelativistic particles, confinement of relativistic runaway electrons [2] has received more attention recently for the modeling of disruptions in reactor-grade fusion devices. For natural disruption conditions without any mitigation scheme, the attainable kinetic energy of runaways is estimated to be on the order of 100 MeV in ITER [3]. Therefore, relativistic treatment of the guiding-center theory is mandatory.

This paper describes a guiding-center model for a runaway electron orbit that includes the effects of the slow equilibrium changes over the resistive timescale. The model considers the induction field produced by nonideal changes in magnetic fluxes. A relatively strong electric field induced during disruptions accelerates hot tails of the electron distribution function and yields significant populations of runaway electrons. Here the toroidal induction field $\boldsymbol{E}_\phi$ is written in terms of the derivative of time of the poloidal flux function ψ_{p}:

$$\boldsymbol{E}_\phi = \left.\frac{\partial \psi_{\mathrm{p}}}{\partial t}\right|_{\boldsymbol{x}} \nabla\phi, \tag{1}$$

where ϕ is a geometrical toroidal angle. The relation between the induction field and the equilibrium evolution becomes clear when we consider the axisymmetric evolution of the poloidal fluxes over the resistive timescale, which was discussed by Riemann *et al.* [4], considering runaway electron generation such that

$$\sigma_\parallel \mu_0 \frac{\partial \psi_{\mathrm{p}}}{\partial t} = \Delta^* \psi_{\mathrm{p}} - \mu_0 R j_r. \tag{2}$$

In (2), Δ^* is the Grad-Shafranov operator

$$\Delta^* = R\frac{\partial}{\partial R}\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}, \tag{3}$$

where (R, Z) denotes the position in the poloidal plane, $\sigma_\parallel$ is the neoclassical conductivity, μ_0 is the vacuum permeability, and j_r is the generated runaway current. The fast temperature drop in an early phase of the disruptions causes a significant increase in the plasma resistivity $\sigma_\parallel^{-1}$, which induces the toroidal voltage above the threshold of runaway generation. Because the generated runaways may cause substantial damage to plasma-facing components and may unacceptably shorten their lifetimes, three-dimensional (3D) orbit calculations for runaway electrons are important, e.g., for evaluating the wall load and for studying physical mechanisms of the runaway generation during disruptions.

In this paper, the guiding-center equations are formulated in Boozer coordinates, which have been widely used in practical simulations of the energetic particle motions. Although relativistic guiding-center equations in magnetic coordinates have appeared in several studies [5–9], we are not aware of any publication that includes the induction field. Although the induction field is normally neglected in energetic-particle simulations, it plays an essential role in accelerating light electrons along the magnetic field and in determining the energy distribution function of runaway electrons generated during disruptions in present and future tokamaks [10].

In our derivation, the radial covariant component of the equilibrium magnetic field is retained, although it is

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often neglected in conventional guiding-center theory [11]. By including this component, we can develop a consistent treatment of the guiding-center orbit for finite-pressure plasmas, which are important, for instance, for high-β_p disruption [12] (where β_p is the poloidal beta value). For this purpose, we applied the Lagrangian formulation that was originally developed by White and Zakahrov [13] for nonrelativistic particle motions in static magnetic fields, to relativistic particle motions in weak time dependent systems. We show that both the canonical and noncanonical equations are obtained in a consistent way for axisymmetric equilibria, including a weak time dependence in the magnetic-flux function.

Section 2 describes the guiding-center Lagrangian formalism for relativistic particles in magnetic coordinates. In Sec. 3, we describe the relativistic guiding-center model in canonical form, following the Lagrangian formalism described in [13]. A transformation of canonical to noncanonical variables is presented in Sec. 4. Finally, conclusions are drawn in Sec. 5. For practical applications in runaway electron generation, we discuss the validity of the ordering with respect to the inductive electric field on the basis of the main characteristics of an ITER-grade disruption.

2. Guiding-Center Lagrangian for Relativistic Electrons

The framework of guiding-center equations is built with the assumption that when it is compared to particle gyromotion, the equilibrium magnetic field varies slowly over spatial and temporal scales. In the following, we normalize quantities with major radius R_0, magnetic field on the axis B_0, and transit time $\omega_t^{-1} = R_0/c$, where c is the speed of light. We introduce the drift-ordering parameter $\epsilon \equiv \omega_t/\omega_c \ll 1$, where $\omega_c = |e|B_0/m$ with particle charge e and mass m.

We begin with a description in Boozer coordinates (s, θ, ζ), where s is the surface label and θ and ζ are the poloidal and toroidal angles, respectively. In axisymmetric tokamaks ($\partial/\partial\zeta = 0$), the equilibrium magnetic field with a nested flux surface can be written in a contravariant representation as

$$\boldsymbol{B} = \psi_t'(s, \epsilon t)\nabla s \times \nabla\theta + \psi_p'(s, \epsilon t)\nabla\zeta \times \nabla s, \tag{4}$$

where $2\pi\psi_t$ and $2\pi\psi_p$ denote the toroidal and poloidal magnetic fluxes inside the magnetic surface, respectively. The prime indicates the derivative with respect to the flux surface label s. The vector potential corresponding to (4) is given by $\boldsymbol{A} = \psi_t\nabla\theta - \psi_p\nabla\zeta$. To include the induction field, we follow [14] and retain an explicit time dependence in the toroidal and poloidal fluxes. This is accomplished by setting $\partial/\partial t \sim O(\epsilon)$, as indicated symbolically in (4). In Boozer coordinates, the equilibrium magnetic field in the covariant representation is

$$\boldsymbol{B} = J_t(s, \epsilon t)\nabla\theta + J_p(s, \epsilon t)\nabla\zeta + \beta_*(s, \theta, \epsilon t)\nabla s, \tag{5}$$

where $(2\pi/\mu_0)J_t$ is the toroidal current flowing inside the magnetic surface, and $(2\pi/\mu_0)J_p$ is the poloidal current flowing outside the magnetic surface (μ_0: with the vacuum permeability). The last term $\beta_*(s, \theta, \epsilon t)$ denotes the radial covariant component of the equilibrium magnetic field; this term is often neglected in the approximate canonical formalism [6].

As is well known, introducing a time dependence in the magnetic-flux function makes the choice of the magnetic surface label s non-trivial in simulations. When we follow the framework of so-called 1.5D transport codes (see [15] and reference therein), it is useful to define the surface label in terms of the toroidal flux function, e.g., such that

$$\rho = \sqrt{\frac{\psi_t}{2B_{\phi 0}}}, \tag{6}$$

where $B_{\phi 0}$ denotes the representative toroidal magnetic field. Recall that for slow evolution of tokamak plasmas over the resistive timescale, a relative motion of the ψ_t and ψ_p contours occurs and the MHD safety factor $q \equiv \mathrm{d}\psi_t/\mathrm{d}\psi_p$ effectively changes due to nonideal effects. Because the strong toroidal field is applied in standard tokamak conditions, the toroidal flux contour moves sufficiently slower than the poloidal flux ones [16]. To simulate such evolutions, it is therefore convenient to choose the reference surface label to be the toroidal flux or its equivalence.

It is also important to consider induced losses of energetic particles resulting from electrostatic and electromagnetic perturbations. Here they are included as perturbed quantities, $\delta\phi$ and $\delta\boldsymbol{A}$. Because energetic electrons are sensitive to details of magnetic-field topologies, losses of relativistic electrons are considered to be mainly due to magnetic perturbations $\delta\boldsymbol{B} = \nabla \times \delta\boldsymbol{A}$. For low beta plasmas, a model of the perturbed vector potential

$$\delta\boldsymbol{A} = V(s, \theta, \zeta)\boldsymbol{B}, \tag{7}$$

is often used. Although it is straightforward to implement more general forms of electromagnetic perturbations, which can be applied to high beta tokamaks or those for representing full electromagnetic waves [17], here we employ (7) in the formulation for simplicity.

To treat weak time dependent systems, let us consider the extended phase space defined by $(t, \boldsymbol{x}, p_\parallel, h)$, where t is the time, $\boldsymbol{x}$ is the guiding-center position, $p_\parallel = \gamma v_\parallel$ is the normalized relativistic parallel momentum, and h is a Hamiltonian variable. The relativistic momenta are defined in terms of the parallel velocity $v_\parallel = \boldsymbol{v} \cdot \boldsymbol{b}$ and the relativistic factor $\gamma = 1/\sqrt{1 - v^2}$. With these definitions, the relativistic guiding-center Lagrangian has the following dimensionless form [18]

$$L\mathrm{d}\tau = \frac{\sigma}{\epsilon}[\boldsymbol{A}_{\mathrm{eq}} + \epsilon\delta\boldsymbol{A} + \epsilon p_\parallel\boldsymbol{b}] \cdot \mathrm{d}\boldsymbol{x} - h\mathrm{d}t - H\mathrm{d}\tau, \tag{8}$$

where $\sigma \equiv \mathrm{sgn}(e)$ denotes the sign of the electric charge. In (8), an independent variable τ is introduced, and the

Hamiltonian has the dimensionless form of $H = \gamma + \delta\phi - h$. By transforming the Lagrangian from the physical space $(t, \boldsymbol{x}, p_\parallel, h)$ to that in Boozer coordinates $(t, s, \theta, \zeta, \rho_c, h)$, where $\rho_c = p_\parallel/(\sigma B) + V$ denotes the canonical parallel gyroradius [19], (8) leads to

$$L\mathrm{d}\tau = \frac{\sigma}{\epsilon}(\psi_t + \epsilon\rho_c J_t)\mathrm{d}\theta + \frac{\sigma}{\epsilon}(-\psi_p + \epsilon\rho_c J_p)\mathrm{d}\zeta, \\ + \sigma\rho_c\beta_* \mathrm{d}s - h\mathrm{d}t - H\mathrm{d}\tau. \tag{9}$$

Similarly, the Hamiltonian is transformed into Boozer coordinates, $H = [1 + (\rho_c - V)^2 B^2 + 2\mu B]^{1/2} + \delta\phi - h$.

3. Canonical form of the Guiding-Center Equations

In early developments of the guiding-center formalism in magnetic coordinates [11], the radial covariant component of the equilibrium magnetic field in (5) was often neglected in reducing the Lagrangian (9) to canonical form. However, the subtlety in neglecting the radial covariant component β_* has been recognized for a long time [13] because β_* does not vanish for finite-pressure plasma or up-down asymmetric configurations. This issue was resolved for axisymmetric equilibria by White and Zakharov [13], where they found an explicit transformation of angle variables to obtain the canonical form of the guiding-center Lagrangian without neglecting β_*. In this paper, we show that their approach can be straightforwardly applied to a relativistic guiding-center model that involves a weak time dependence in the magnetic-flux functions.

To cast (9) into a canonical form, let us introduce new toroidal and poloidal angles, ζ_c and θ_c, such that

$$\zeta_c = \zeta + F(s, \theta, \epsilon t), \tag{10a}$$
$$\theta_c = \theta. \tag{10b}$$

The generating function $F(s, \theta)$, which also involves a weak time dependence on the order of ϵ, is here defined by [13]

$$F(s, \theta, \epsilon t) = \int_0^s \frac{\beta_*(s, \theta, \epsilon t)}{J_p(s, \epsilon t)}\mathrm{d}s. \tag{11}$$

Hence we obtain the total derivative of ζ as

$$\dot{\zeta} = \dot{\zeta}_c - \frac{\partial F}{\partial t}\dot{t} - \frac{\beta_*}{J_p}\dot{s} - \frac{\partial F}{\partial \theta}\dot{\theta},$$

where the dot indicates derivative with respect to τ. The essence of (10) is to choose the generating function such that the transformation prevents the poloidal angle from being changed and modifies an angle variable only in the symmetric direction of the equilibrium magnetic field, which is the toroidal direction in case of axisymmetric tokamak geometry. As it can be shown, a similar transformation method to canonical variables becomes intrinsically implicit in systems having no symmetry direction, such as stellarator and helical devices. For such systems, nonlinear equations needs to be solved by iteration for obtaining the canonical variables.

Inserting the new poloidal and toroidal angles θ_c and ζ_c into the Lagrangian and applying the formula of partial integrals with respect to the surface label s, $fg = \int f'g\mathrm{d}s + \int fg'\mathrm{d}s$, we obtain a canonical form of the relativistic guiding-center Lagrangian:

$$L = p_\theta^c \dot{\theta}_c + p_\zeta^c \dot{\zeta}_c - h_c \dot{t} + \dot{S} - H. \tag{12}$$

Here, the poloidal and toroidal canonical momenta p_θ^c and p_ζ^c and a new Hamiltonian variable h_c are defined by

$$p_\theta^c = \frac{\sigma}{\epsilon}\left[\psi_t + \int \psi_p' \frac{\partial F}{\partial \theta}\mathrm{d}s + \epsilon\rho_c\left(J_t - J_p\frac{\partial F}{\partial \theta_c}\right)\right], \tag{13a}$$

$$p_\zeta^c = \frac{\sigma}{\epsilon}(-\psi_p + \epsilon\rho_c J_p), \tag{13b}$$

$$h_c = h + p_\zeta^c\frac{\partial F}{\partial t} + \frac{\partial S}{\partial t}. \tag{13c}$$

Because in (12), the derivative of the gauge function

$$S(s, \theta, \epsilon t) = \int_0^s \psi_p(s, \epsilon t)\frac{\beta_*(s, \theta, \epsilon t)}{J_p(s, \epsilon t)}\mathrm{d}s, \tag{14}$$

does not affect guiding-center motion, $\dot{S}$ can be eliminated from the Lagrangian. Comparing (13) with its counterpart in [13], the Hamiltonian variable h is transformed to h_c in the present case, which manifests a weak time dependence in the equilibrium magnetic field. Finally, we obtain the equation of motion in the form of the Hamilton's equation from (12):

$$\dot{t} = 1, \quad \dot{h}_c = \frac{\partial H}{\partial t},$$
$$\dot{\theta}_c = \frac{\partial H}{\partial p_\theta^c}, \quad \dot{p}_\theta^c = -\frac{\partial H}{\partial \theta_c},$$
$$\dot{\zeta}_c = \frac{\partial H}{\partial p_\zeta^c}, \quad \dot{p}_\zeta^c = -\frac{\partial H}{\partial \zeta_c}, \tag{15}$$

where the Hamiltonian in canonical coordinates is given by

$$H = \sqrt{1 + \left[\frac{1}{J_p}\left(\frac{\psi_p}{\epsilon} + \sigma p_\zeta^c\right) - V\right]^2 B^2 + 2\mu B} \\ + \delta\phi - h_c - p_\zeta^c\frac{\partial F}{\partial t} - \frac{\partial S}{\partial t}.$$

Note here that when we neglect potential $(\delta\phi)$ and magnetic (V) fluctuations, the Hamiltonian does not involve any dependence on the toroidal angle ζ_c for axisymmetric systems. It is therefore clear from (15) that toroidal canonical momentum is exactly conserved.

4. Noncanonical Form of the Guiding-Center Equations

While the canonical form presented in Sec. 3 has theoretical merit in analytic treatments, e.g., to understand the conserving properties of the guiding-center motion, the noncanonical form is useful for practical purposes of numerical implementation. Noncanonical equations of motion are derived from Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\partial L}{\partial \dot{Z}_i}\right) - \frac{\partial L}{\partial Z_i} = 0, \quad (i = t, s, \theta, \zeta, \rho_\mathrm{c}, h). \tag{16}$$

Using the guiding-center Lagrangian (9), we obtain Euler-Lagrange equations for the noncanonical variables in an explicit way:

$$\begin{aligned}
&-\frac{\partial H}{\partial s}\dot{s} - \frac{\partial H}{\partial \theta}\dot{\theta} - \frac{\partial H}{\partial \zeta} - \frac{\partial H}{\partial \rho_\mathrm{c}} - \sigma\left[\frac{\partial \psi_\mathrm{t}}{\partial t}\dot{\theta} - \frac{\partial \psi_\mathrm{p}}{\partial t}\dot{\zeta}\right] = 0,\\
&\frac{\sigma}{\epsilon}\left(\psi_\mathrm{t}'\dot{s} + \epsilon\frac{\partial \psi_\mathrm{t}}{\partial t}\dot{t} + \epsilon\dot{\rho}_\mathrm{c}J_\mathrm{t} + \epsilon\rho_\mathrm{c}J_\mathrm{t}'\dot{s}\right)\\
&\qquad -\left[\sigma\rho_\mathrm{c}\frac{\partial \beta_*}{\partial \theta}\dot{s} - \frac{\partial H}{\partial \theta}\dot{t}\right] = 0,\\
&\frac{\sigma}{\epsilon}\left(-\psi_\mathrm{p}'\dot{s} - \epsilon\frac{\partial \psi_\mathrm{p}}{\partial t}\dot{t} + \epsilon\dot{\rho}_\mathrm{c}J_\mathrm{p} + \epsilon\rho_\mathrm{c}J_\mathrm{p}'\dot{s}\right) + \frac{\partial H}{\partial \zeta}\dot{t} = 0,\\
&\sigma\dot{\rho}_\mathrm{c}\beta_* + \sigma\rho_\mathrm{c}\frac{\partial \beta_*}{\partial \theta}\dot{\theta} - \left[\frac{\sigma}{\epsilon}(\psi_\mathrm{t}' + \epsilon\rho_\mathrm{c}J_\mathrm{t}')\dot{\theta}\right.\\
&\qquad \left. + \frac{\sigma}{\epsilon}(-\psi_\mathrm{p}' + \epsilon\rho_\mathrm{c}J_\mathrm{p}')\dot{\zeta} - \frac{\partial H}{\partial s}\dot{t}\right] = 0,\\
&\sigma J_\mathrm{t}\dot{\theta} + \sigma J_\mathrm{p}\dot{\zeta} + \sigma\beta_*\dot{s} - \frac{\partial H}{\partial \rho_\mathrm{c}}\dot{t} = 0.
\end{aligned} \tag{17}$$

By inverting (17) with respect to $(\dot{t}, \dot{s}, \dot{\theta}, \dot{\zeta}, \dot{\rho}_\mathrm{c}, \dot{h})$, we obtain noncanonical forms of the guiding-center equations; this would be the most straightforward method. Nonetheless, to check consistency, we consider a different approach, a direct transformation of canonical to noncanonical variables using (13).

Let us consider the Poisson brackets of arbitrary phase-space functions f and g in terms of the canonical variables, which are given in diagonalized form:

$$\begin{aligned}
\{f, g\} = &\frac{\partial f}{\partial \theta_\mathrm{c}}\frac{\partial g}{\partial p_\theta^\mathrm{c}} - \frac{\partial f}{\partial p_\theta^\mathrm{c}}\frac{\partial g}{\partial \theta_\mathrm{c}} + \frac{\partial f}{\partial \zeta_\mathrm{c}}\frac{\partial g}{\partial p_\zeta^\mathrm{c}}\\
&-\frac{\partial f}{\partial p_\zeta^\mathrm{c}}\frac{\partial g}{\partial \zeta_\mathrm{c}} - \frac{\partial f}{\partial t}\frac{\partial g}{\partial h_\mathrm{c}} + \frac{\partial f}{\partial h_\mathrm{c}}\frac{\partial g}{\partial t}.
\end{aligned} \tag{18}$$

Using (13) and (18), we can calculate the Poisson brackets $\{Z_i, Z_j\}$ with respect to each pair of the noncanonical variables $Z_i \equiv (t, s, \theta, \zeta, \rho_\mathrm{c}, h)$. For this, we employ the transformation rule of Poisson brackets $Z_i \to \bar{Z}_i$:

$$\{\bar{Z}_i, \bar{Z}_j\} = \sum_{m,n}\frac{\partial \bar{Z}_i}{\partial Z_m}\{Z_m, Z_n\}\frac{\partial \bar{Z}_j}{\partial Z_n}. \tag{19}$$

After some lengthy manipulations, we obtain a set of Poisson brackets with respect to each pair of noncanonical variables as follows:

$$\begin{aligned}
&\{t, s\} = 0, \quad \{t, \theta\} = 0, \quad \{t, \zeta\} = 0,\\
&\{t, \rho_\mathrm{c}\} = 0, \quad \{t, h\} = -1,\\
&\{s, \theta\} = -\epsilon\frac{J_\mathrm{p}}{\sigma\mathcal{D}}, \quad \{s, \zeta\} = \epsilon\frac{J_\mathrm{t}}{\sigma\mathcal{D}}, \quad \{s, \rho_\mathrm{c}\} = 0,\\
&\{s, h\} = \frac{\epsilon}{\mathcal{D}}\left(J_\mathrm{t}\frac{\partial \psi_\mathrm{p}}{\partial t} + J_\mathrm{p}\frac{\partial \psi_\mathrm{t}}{\partial t}\right) + O(\epsilon^2),\\
&\{\theta, \zeta\} = -\epsilon\frac{\beta_*}{\sigma\mathcal{D}}, \quad \{\theta, \rho_\mathrm{c}\} = -\frac{1}{\sigma\mathcal{D}}(-\psi_\mathrm{p}' + \epsilon\rho_\mathrm{c}J_\mathrm{p}'),\\
&\{\theta, h\} = \epsilon\frac{\beta_*}{\mathcal{D}}\frac{\partial \psi_\mathrm{p}}{\partial t} + O(\epsilon^2),\\
&\{\zeta, \rho_\mathrm{c}\} = \frac{1}{\sigma\mathcal{D}}\left[\psi_\mathrm{t}' + \epsilon\rho_\mathrm{c}\left(J_\mathrm{t}' - \frac{\partial \beta_*}{\partial \theta}\right)\right],\\
&\{\zeta, h\} = -\epsilon\frac{\beta_*}{\mathcal{D}}\frac{\partial \psi}{\partial t} + O(\epsilon^2),\\
&\{\rho_\mathrm{c}, h\} = -\frac{1}{\mathcal{D}}(-\psi_\mathrm{p}' + \epsilon\rho_\mathrm{c}J_\mathrm{p}')\frac{\partial \psi_\mathrm{t}}{\partial t}\\
&\qquad -\frac{1}{\mathcal{D}}\left[\psi_\mathrm{t}' + \epsilon\rho_\mathrm{c}\left(J_\mathrm{t}' - \frac{\partial \beta_*}{\partial \theta}\right)\right]\frac{\partial \psi_\mathrm{p}}{\partial t} + O(\epsilon^2),
\end{aligned} \tag{20}$$

where $\mathcal{D}$ denotes the Jacobian with respect to the phase-space flow of guiding-center motion

$$\begin{aligned}
\mathcal{D} = &\psi_\mathrm{p}'J_\mathrm{t} + \psi_\mathrm{t}'J_\mathrm{p}\\
&+\epsilon\rho_\mathrm{c}\left(J_\mathrm{p}J_\mathrm{t}' - J_\mathrm{t}J_\mathrm{p}' - J_\mathrm{p}\frac{\partial \beta_*}{\partial \theta}\right).
\end{aligned} \tag{21}$$

In (20), we indicate in a symbolic manner, the second-order terms $O(\epsilon^2)$ contained in the transformed Poisson brackets. In deriving (20), correction terms appear due to the weak time dependence of the generating (F) and gauge (S) functions, but they appear only in the second order. Hence, they can be neglected consistently in deriving the guiding-center equations of the first order of the gyroradii, which are used in standard numerical simulations. Finally, the noncanonical equations of motion can be obtained using (20) in the form

$$\dot{Z}_i = \{Z_i, H\} = \{Z_i, Z_j\}\frac{\partial H}{\partial Z_j}, \tag{22}$$

where $Z_i = (t, s, \theta, \zeta, \rho_c, h)$. Writing (22) in an explicit manner, we obtain

$$\dot{s} = \frac{J_\mathrm{p}}{\mathcal{D}}\left(-\frac{\epsilon}{\sigma}\frac{\partial H}{\partial \theta} - \epsilon\frac{\partial \psi_\mathrm{t}}{\partial t}\right) - \frac{J_\mathrm{t}}{\mathcal{D}}\left(-\frac{\epsilon}{\sigma}\frac{\partial H}{\partial \zeta} + \epsilon\frac{\partial \psi_\mathrm{p}}{\partial t}\right), \tag{23a}$$

$$\begin{aligned}
\dot{\theta} = &\epsilon\frac{J_\mathrm{p}}{\sigma\mathcal{D}}\frac{\partial H}{\partial \psi_\mathrm{p}} - \frac{1}{\sigma\mathcal{D}}(-1 + \epsilon\rho_\mathrm{c}J_\mathrm{p}')\frac{\partial H}{\partial \rho_\mathrm{c}}\\
&-\epsilon\frac{\beta_*}{\sigma\mathcal{D}}\frac{\partial H}{\partial \zeta} + \epsilon\frac{\beta_*}{\mathcal{D}}\frac{\partial \psi_\mathrm{p}}{\partial t},
\end{aligned} \tag{23b}$$

$$\begin{aligned}
\dot{\zeta} = &-\epsilon\frac{J_\mathrm{t}}{\sigma\mathcal{D}}\frac{\partial H}{\partial s} + \frac{1}{\sigma\mathcal{D}}\left[q + \epsilon\rho_\mathrm{c}\left(J_\mathrm{t}' - \frac{\partial \beta_*}{\partial \theta}\right)\right]\frac{\partial H}{\partial \rho_\mathrm{c}}\\
&+\epsilon\frac{\beta_*}{\sigma\mathcal{D}}\frac{\partial H}{\partial \theta} + \epsilon\frac{\beta_*}{\mathcal{D}}\frac{\partial \psi_\mathrm{t}}{\partial t},
\end{aligned} \tag{23c}$$

$$\begin{aligned}
\dot{\rho}_\mathrm{c} = &\frac{1}{\epsilon\mathcal{D}}\left[q + \epsilon\rho_\mathrm{c}\left(J_\mathrm{t}' - \frac{\partial \beta_*}{\partial \theta}\right)\right]\left(-\frac{\epsilon}{\sigma}\frac{\partial H}{\partial \zeta} + \epsilon\frac{\partial \psi_\mathrm{p}}{\partial t}\right)\\
&-\frac{1}{\epsilon\mathcal{D}}(-1 + \epsilon\rho_\mathrm{c}J_\mathrm{p}')\left(-\frac{\epsilon}{\sigma}\frac{\partial H}{\partial \theta} - \epsilon\frac{\partial \psi_\mathrm{t}}{\partial t}\right).
\end{aligned} \tag{23d}$$

The terms containing the radial covariant component β_* in (23) are normally neglected in the approximate canonical theory. We can check that the expressions in (23) are consistent with what is obtained by inverting the Euler-Lagrange equations (17) up to the order of ϵ. Equations (23) describe the effects of slow equilibrium change and the associated electron acceleration in toroidal geometry. In addition, they also involve the radial, poloidal, and toroidal drift terms due to the induction field, $\partial\psi_p/\partial t$ and $\partial\psi_t/\partial t$, which are related to the inward motion of trapped particles in the presence of the toroidal electric field [20]. In (23), relativistic corrections appear in the derivatives of the Hamiltonian, which are given such that

$$\left.\frac{\partial H}{\partial s}\right)_{t,\theta,\zeta,\rho_c,h} = \frac{\delta}{\gamma}\frac{\partial B}{\partial s} - \frac{\rho_\parallel B^2}{\gamma}\frac{\partial V}{\partial s} + \frac{\partial\delta\phi}{\partial s}, \tag{24a}$$

$$\left.\frac{\partial H}{\partial \theta}\right)_{t,s,\zeta,\rho_c,h} = \frac{\delta}{\gamma}\frac{\partial B}{\partial \theta} - \frac{\rho_\parallel B^2}{\gamma}\frac{\partial V}{\partial \theta} + \frac{\partial\delta\phi}{\partial \theta}, \tag{24b}$$

$$\left.\frac{\partial H}{\partial \zeta}\right)_{t,s,\theta,\rho_c,h} = - \frac{\rho_\parallel B^2}{\gamma}\frac{\partial V}{\partial \zeta} + \frac{\partial\delta\phi}{\partial \zeta}, \tag{24c}$$

$$\left.\frac{\partial H}{\partial \rho_c}\right)_{t,s,\theta,\zeta,h} = \frac{\rho_\parallel B^2}{\gamma}, \tag{24d}$$

with $\delta = \mu + \rho_\parallel^2 B$.

For numerical simulations, the other sets of noncanonical variables such as those including the parallel gyroradius $(t, s, \theta, \zeta, \rho_\parallel, h)$ or the parallel momentum $(t, s, \theta, \zeta, p_\parallel, h)$ are often employed, where $\rho_\parallel = \rho_c - V$. Such a transformation to other sets of noncanonical coordinates is straightforward with the help of the table of the Poisson brackets in (20).

5. Discussion and Conclusions

Guiding-center equations for relativistic particles are derived that consider the induction field produced by slow equilibrium changes over the resistive timescale. We retain an explicit time dependence in the equilibrium magnetic field, following [14], by writing $\partial/\partial t \sim O(\epsilon)$. Our formulation is summarized in Fig. 1. We begin with the first-order guiding-center Lagrangian in the physical space $(t, \boldsymbol{x}, p_\parallel, h)$ and express it in terms of Boozer coordinates (s, θ, ζ) and the canonical parallel gyroradius ρ_c. By applying the transformation of the toroidal angle $\zeta \to \zeta_c$, we obtain a set of canonical variables $(t, \theta_c, \zeta_c, h_c, p_\theta^c, p_\zeta^c)$ in the extended phase space. Note that the guiding-center equations (23) obtained from the transformation of the Poisson brackets from canonical variables to noncanonical ones are identical to what is obtained by directly inverting the Euler-Lagrange equations in (17). This illustrates, within the order of accuracy considered here, the equivalence between the canonical (15) and noncanonical forms (23) even with their extension to weak time dependent systems. As is seen from the Poisson brackets in (20), since the correction terms containing the generating (F) and gauge (S) functions appear only in the second order, they can be neglected consistently in deriving the first-order guiding center equations. In this formulation, because no approximations on the magnitude of β_* are applied, both the canonical and noncanonical guiding-center equations obtained here appropriately recover the original ones that employs the physical space variables $(t, \boldsymbol{x}, h)$ [18] for finite-pressure plasmas.

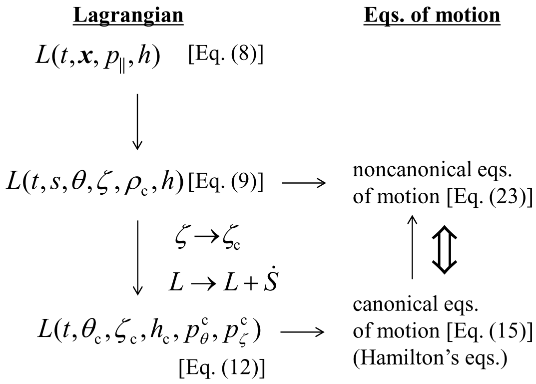

Fig. 1 Summary of the derivation in this paper. The Lagrangian with respect to noncanonical variables $(t, s, \theta, \zeta, \rho_c, h)$ in Boozer coordinates is transformed into canonical variables by means of the transformation of the toroidal angle $\zeta \to \zeta_c$, which yields the gauge transformation of the Lagrangian, $L \to L + \dot{S}$ in (12). The canonical and noncanonical forms of the guiding-center equations obtained are equivalent to each other up to the first order in the drift-ordering parameter.

The guiding-center model derived here assumes that the induction field due to changes in the magnetic-flux functions $\psi_t(s, \epsilon t)$ and $\psi_p(s, \epsilon t)$ is first order. We briefly discuss the validity of this assumption for simulations of runaway generation in tokamak experiments. For ITER, the in-plasma electric field is evaluated to be 38 V/m [3] for the disruption of a plasma with a toroidal current I_p = 15 MA. This is much weaker than the Dreicer field threshold,

$$E_D = \frac{n_e e^3 \ln\Lambda}{4\pi\epsilon_0^2 T_e}, \tag{25}$$

where n_e is the electron density, T_e is the temperature in energy units, $\ln\Lambda$ is the Coulomb logarithm, and ϵ_0 is the vacuum permittivity. If the electric field is larger than the Dreicer electric field, $E_\parallel > E_D$, bulk electrons are accelerated into the runaway region in momentum space for which the $E_\parallel$-acceleration exceeds the collisional friction force against thermal electrons. Nevertheless, no such strong Dreicer generation is anticipated in tokamak disruptions because of its high density; typical $E_\parallel$ values for tokamak disruptions are in the intermediate regime such that $E_c \lesssim E_\parallel \ll E_D$, for which only a fraction of thermal electrons, i.e., a hot tail, becomes runaway. Here

$$E_c = \frac{n_e e^3 \ln\Lambda}{4\pi\epsilon_0^2 m_e c^2}, \tag{26}$$

is the critical electric field that defines the threshold of runaway generation. Because of the relativistic constraint

[21], absolutely no runaway occurs if $E_{\|} < E_{\rm c}$. For the above $E_{\|}$ value, the electron acceleration time $t_{\rm acc} = m_{\rm e}c/eE_{\|}$ is on the order of 10^{-4}–10^{-5}, which is much slower than the electron transit time $2\pi/\omega_{\rm t} = 2\pi R_0/c \sim 10^{-7}$. Therefore, the condition that $t_{\rm acc}^{-1}/\omega_{\rm t} \ll 1$ postulates the use of weak induction field ordering. Note here that because the induction field is fairly weak in the above comparison, setting their effect to second-order, i.e., $\partial/\partial t \sim O(\epsilon^2)$, may also be appropriate. Even though modifications to the cross-field drift $\dot{s}$, $\dot{\theta}$, and $\dot{\zeta}$ are negligible, the toroidal acceleration term $\partial\psi_{\rm p}/\partial t$ still has an essential contribution in determining rapid parallel motion of relativistic electrons. Neglecting this term limits the applications of relativistic guiding-center formalisms, e.g., to evaluate the energy distribution of runaway electrons. The latter is an important part in modeling runaway electron behavior during tokamak disruptions, e.g., for predicting the energy flux flowing from the core region to the first wall.

For actual simulations, appropriate models for evaluating the induction field are necessary. One candidate is to use 1.5D transport codes (e.g., DINA [22], TASK [23], and TOPICS [15]). Following tokamak transport simulations using these codes, the runaway orbit is calculated for a given MHD equilibrium with the magnetic coordinates, and the latter is updated after an appropriate time-step over the slow resistive timescale under the influence of the external coil current. We mention that the weak induction-field ordering used here is appropriate for building such a framework of integrated simulations of disruption and runaway electrons. In practice, the feature that the fast MHD timescale is eliminated in transport simulations is important to carry out long-term simulations covering the whole disruption lifetime; such lifetimes are comparable to several hundred milliseconds or one second in an ITER-grade device.

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