On a Nonlinear Dispersion Effect of Geodesic Acoustic Modes

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The nonlinear dispersion relation of the geodesic acoustic modes (GAMs) is investigated for tokamaks with a high safety factor and low magnetic shear. We focus on the Reynolds stress as a nonlinearity, which is truncated at the third order of the GAM amplitude. The real frequency of the GAM is modified according to the phase of the nonlinear force acting on the GAM, which depends on the turbulence decorrelation rate. The nonlinear frequency shift is much larger than that from the finite gyro-radius effects in the linear theory, when the poloidal turbulent $E \times B$ velocities are comparable to the diamagnetic drift velocity. Under such circumstances, the group velocity is strongly enhanced and becomes comparable with the radial phase velocity. In addition, the magnitude of the nonlinear effects is also evaluated using experimental parameters.

Keywords: geodesic acoustic mode, zonal flow, drift wave, turbulence, dispersion relation, wave propagation

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1. Introduction

Zonal flows have attracted much attention in the research of magnetically confined plasmas [1]. A geodesic acoustic mode (GAM) is a type of zonal flow with a finite real frequency, which is due to the geodesic curvature of a toroidal magnetic field [2]. Coupling between the GAMs and the turbulence has been observed in many toroidal devices [3–6], and can even affect the level of turbulence transport [7, 8]. The excitation of GAMs due to the nonlinear coupling with the turbulence with a broad spectrum [9, 10], and with coherent drift waves [11, 12] has been theoretically studied. The nonlinear theory of the GAMs has been developed to include higher order nonlinear coupling such as the generation of the second harmonics [13–15], and the coupling with the zero frequency zonal flows (ZFs) [16–19]. Recently, in a process called GAM channeling, the ion heating rate of the GAMs driven by energetic particles was reported to be nonnegligible [20]. Evidently, the various important roles of the GAMs are increasingly being recognized.

The GAMs have nonlocal effects on plasmas, because they form radial eigenmodes, and propagate radially with a finite group velocity. Radial eigenmodes have been observed experimentally [21–23], and explained theoretically [24–27]. The radial group velocities of the GAMs have been studied using the linear dispersion relation based on the gyro-kinetic formalism [28], and the ratio between the group and phase velocities has been predicted as $v_g/v_p \sim q_r^2\rho_s^2 \ll 1$, where v_g, v_p, q_r, ρ_s are the group and phase velocities, the radial wavenumber of the GAM, and the ion gyro radius based on the sound velocity, respectively [29, 30]. Nonlinear simulations of the GAMs have pointed out that the group velocity is strongly enhanced by the nonlinearity as $v_g/v_p \sim 100q_r^2\rho_s^2$ [33]. Attempts to measure the propagation characteristics of the GAMs in experiments have begun [31, 32]. Hence, a physical understanding of the nonlinear effect on the GAM propagation is in high demand. The real frequency of the GAM is also important for a novel method to measure the ion mass, called GAM spectroscopy, which is based on the GAM frequency and the radial eigenmode [34]. There are many studies on the frequency shift due to linear influences such as plasma shaping [35, 36], the finite ion gyro radius [27–30, 37], and impurities [38, 39]. However, the nonlinear frequency shift has not been understood, so the nonlinear frequency shift should be investigated theoretically.

Nonlinearities such as the Reynolds stress can drive the GAMs [1]. For a high safety factor and low magnetic shear, the Reynolds stress becomes important compared with the dynamical shearing induced Winsor drive [9]. (For example, such plasmas are obtained with a reversed shear tokamak configuration.) Therefore, it is worthwhile to investigate the nonlinear characteristics of the GAMs associated with the Reynolds stress. In this study, the nonlinear dispersion relation of the GAM is theoretically investigated in a simple magnetic geometry. Based on the wave kinetic theory, the Reynolds stress is evaluated, and

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the nonlinear dispersion relation, which includes the nonlinearities of the drift wave turbulence and the GAM, is derived. Analytical expressions for the nonlinear frequency shift and the propagation velocity are obtained and the importance of the nonlinear effect on the frequency shift and the propagation velocity is elucidated by evaluating them with realistic parameters.

2. Model

We consider a tokamak plasma with drift wave turbulence and GAMs. The turbulence is fully developed because of the background gradient and the GAMs are driven by coupling to the turbulence. The zero frequency zonal flow (ZF) is not considered here. In this section, a model for analyzing the nonlinear dispersion relation of the GAMs is introduced, focusing on nonlinearity of the Reynolds stress.

2.1 Model equation for GAM dynamics

The basic equations for the nonlinear GAM dynamics are presented in this subsection. We consider a high aspect ratio, circular cross-section toroidal plasma. In this study, we focus on plasmas with high safety factor $q \gg 1$ and weak magnetic shear, where the GAMs can be excited. The toroidal coordinates (r, θ, ζ) are used, where $\nabla r, \nabla\theta, \nabla\zeta$ are in the radial, poloidal and toroidal directions, respectively. The fluctuations of the GAM, which have toroidal symmetry, are governed by the vorticity, parallel momentum, and continuity equations [40],

$$\partial_t U + \alpha\partial_r\langle n\sin\theta\rangle + \gamma_{\mathrm{damp}}U = -\partial_r^2\langle\Pi_{r\theta}\rangle, \tag{1a}$$

$$\partial_t v_\parallel - \mu\partial_\perp^2 v_\parallel + \partial_\parallel n = 0, \tag{1b}$$

$$\partial_t n - 2\alpha^{-1}\partial_r^{-1}U\sin\theta + \partial_\parallel v_\parallel = 0. \tag{1c}$$

Here, U, $v_\parallel$, and n are the toroidal component of the vorticity averaged over the magnetic surface and normalized to the drift wave frequency ω_*, the parallel velocity normalized to the sound velocity c_s, and the density perturbation normalized to the equilibrium density, respectively. The units for time and space are chosen as the inverse of the typical GAM frequency $\omega_G = \sqrt{2}c_s/R$, where R is the major radius, and the ion sound gyro-radius ρ_s. A flux surface average is represented as $\langle\cdots\rangle = (2\pi)^{-1}\oint\cdots d\theta$. The parallel derivative $\partial_\parallel$ is $(\sqrt{2}q)^{-1}\partial_\theta$. The relevant flux surface averaged Reynolds stress component is $\langle\Pi_{r\theta}\rangle$. The dimensionless parameter α is $\omega_G R/(\omega_*\rho_s)$. The coefficient γ_{damp} in the vorticity equation represents the collisional damping rate, and the term $\mu\partial_\perp^2 v_\parallel$ in the parallel momentum equation denotes the parallel turbulent viscosity and the Landau damping, which are treated as parameters. In these model equations, we focus on the effect of the nonlinearity of the Reynolds stress on the dispersion relation, whereas other nonlinear effects such as the parallel nonlinearity are neglected. We also neglect the driving force of the GAM by the modulation of turbulent particle transport [9], which vanishes in the limit of weak magnetic shear.

We consider a stationary state in which a GAM propagates outward with monochromatic wavenumber $p > 0$,

$$U = u e^{-i\omega t + ipr} + c.c., \tag{2}$$

where u is the Fourier component of the GAM vorticity, and is chosen to be real. For the parallel velocity and density, only the linear response to the vorticity needs to be considered. Eliminating n and $v_\parallel$ from Eqs. (1a)-(1c), the nonlinear dispersion relation becomes

$$i\Omega u + p^2\langle\Pi_{r\theta}(\omega, p)\rangle = 0, \tag{3a}$$

$$\begin{aligned}\Omega &= \omega + \left\{-\omega + \frac{1}{2q^2(\omega + i\mu p_\perp^2)}\right\}^{-1} + i\gamma_{\mathrm{damp}} \\ &\approx \omega - \frac{2q^2\omega}{2q^2\omega^2 - 1} + i\left(\frac{\mu p_\perp^2}{2q^2} + \gamma_{\mathrm{damp}}\right),\end{aligned} \tag{3b}$$

where $\langle\Pi_{r\theta}(\omega, p)\rangle$ is the (ω, p) component of the Reynolds stress, and $p_\perp$ is the perpendicular wavenumber. We assume the terms $\mu p_\perp^2$, γ_{damp} are small, and only keep the lowest order of the linear damping rates, $\mu p_\perp^2$, γ_{damp}, in order to keep the argument transparent. This assumption does not conflict with the situation $q \gg 1$, because the Landau damping is a decreasing function of the safety factor [28]. The imaginary part of Ω is evaluated by using the linear GAM frequency, which is shown below. In the absence of the Reynolds stress, the dispersion relation, Eq. (3a), becomes $\Omega = 0$, which reproduces the linear GAM dispersion relation as

$$\omega = \omega_L - i\frac{q^2}{1 + 2q^2}\left(\frac{\mu p_\perp^2}{2q^2} + \gamma_{\mathrm{damp}}\right), \tag{4}$$

where ω_L is the linear GAM frequency, defined as $\omega_L = \sqrt{1 + 1/(2q^2)}$. In the derivation of Eq. (4), we assume $\mathrm{Im}\omega \ll 1$. The Reynolds stress changes the GAM dispersion relation in accordance with Eq. (3a). It is evaluated in the next subsection.

2.2 Evaluation of Reynolds stress

The flux surface averaged Reynolds stress from the drift wave turbulence is given by

$$\langle\Pi_{r\theta}\rangle(r, t) = \int \frac{k_r k_\theta}{(1 + k_\perp^2)^2} N_k(r, t) d^2k, \tag{5}$$

where k_r, k_θ, and $k_\perp$ are the radial, poloidal and perpendicular components of the wavenumbers of the drift wave turbulence, respectively. The k-Fourier component of the action of the turbulence is denoted as $N_k(r, t) = (1 + k_\perp^2)^2|\tilde{\phi}_k(r, t)|^2$, where $|\tilde{\phi}_k(r, t)|$ is the envelope of the k-Fourier component of the turbulence potential, normalized to $\omega_G^{-2}B^{-2}\rho_s^{-4}$. Here, the variable k of $N_k(r, t)$ represents the fast variation of the drift wave fluctuations, and r represents the slow variations induced by the GAM. The background turbulence has a short correlation time, so that the Reynolds stress has a component that rapidly changes in time, the noise term. We neglect the noise effect, which is beyond the scope of this study. Equation (5) can be calculated from the modulation component of $N_k(r, t)$, which can be evaluated on the basis of the wave kinetic equation [41]. The action of the turbulence is modulated by the spatial gradient of the Doppler shift of the turbulence frequency by the GAM [1]. Here, we neglect the effect of turbulence group velocity. The relevant Fourier component of the Reynolds stress can be expanded in terms of the GAM amplitude u as [19]

$$-p^2\langle \varPi_{r\theta}(\omega, p)\rangle = \varGamma_1 u - \varGamma_2 u^3 + \cdots , \tag{6}$$

where $\varGamma_1$ comes from the quasi-linear response of the turbulence to u, and $\varGamma_2$ is the higher order response of the turbulence, which can be expressed as

$$\varGamma_1 = c_1\left(1 - i\frac{\omega}{\varDelta\omega}\right)^{-1} \tag{7a}$$

$$\varGamma_2 = c_2\left\{2\left(1+\frac{\omega^2}{\varDelta\omega^2}\right)^{-1}\left(1 - i\frac{\omega}{\varDelta\omega}\right)^{-1} + \left(1 - i\frac{\omega}{\varDelta\omega}\right)^{-2}\left(1 - i\frac{2\omega}{\varDelta\omega}\right)^{-1}\right\}. \tag{7b}$$

The nonlinear decorrelation rate of the turbulence is denoted by $\varDelta\omega$, and the normalized coefficients that depend on the spectrum of the turbulence c_1 and c_2 are defined as

$$c_1 = p^2\int \frac{k_\theta^2}{(1+k_\perp^2)^2\varDelta\omega_k}N_k^{(0)}\mathrm{d}^2k, \tag{8a}$$

$$c_2 = p^2\omega_*^2\int \frac{4k_\theta^4}{(1+k_\perp^2)^3\varDelta\omega_k^3}N_k^{(0)}\mathrm{d}^2k. \tag{8b}$$

Here $N_k^{(0)}$ is the action without modulation by the GAMs, and $\varDelta\omega_k$ is the nonlinear damping rate of the k-Fourier component of the drift waves. The coefficients c_1 and c_2 are positive so that the first and second terms in the right hand side of Eq. (6) represent a quasilinear driving force and a nonlinear stabilization force of the GAM, respectively. As in Eq. (6), the expansion of the Reynolds stress has only the terms of odd powers of u in this case, because the largest contribution in the higher order nonlinearity comes from the back interaction of generating the second harmonics. If the coupling between the GAMs and the ZF exists, the terms with even powers of u appear in the expansion of the Reynolds stress [19], which is expected to change the nonlinear dispersion relation.

In this study, we assume strong turbulence, and treat the amplitude of the turbulence $|\tilde{\phi}_k|$ as being of order $\omega_* k_\perp^{-2}$. Moreover, we take the toroidal vorticity of the turbulence to be comparable to the drift wave frequency ω_*, and assume the GAM amplitude to be small ($u^2 \ll 1$), so that we can truncate the Reynolds stress after the third order of u. Care must be taken with the ordering of c_1: it can be estimated from the definition Eq. (8a) as

$$c_1 \sim \frac{p^2k_\theta^2\omega_*}{k_\perp^4}, \tag{9}$$

and depends on the choice of $p/k_\perp$. In this study, we concentrate on $c_1 \sim (\omega_*)^{-1} < 1$, i.e., we are interested in the range of wavenumbers $p/k_\perp \sim (\omega_*)^{-1}$. The frequency shift and the propagation velocity depend strongly on c_1, as shown in the next section. The strength of the nonlinear damping rate of the GAM is characterized by c_2, whose ordering is $c_2/c_1 \sim 4k_\theta^2$. We assume c_2/c_1 is of order unity.

3. Nonlinear Dispersion Relation

In this section, we derive the GAM dispersion relation, including the nonlinear effects, and investigate the real frequency and characteristics of propagation. The nonlinear dispersion relation is obtained by substituting the expression for the Reynolds stress Eq. (6) into Eq. (3a) as

$$-\mathrm{Im}\varOmega + \mathrm{Re}[\varGamma_1 - \varGamma_2 u^2] + i\left\{\mathrm{Re}\varOmega + \mathrm{Im}[\varGamma_1 - \varGamma_2 u^2]\right\} = 0. \tag{10}$$

The real and imaginary parts of Eq. (10) determine the saturation amplitude and nonlinear frequency of the GAM. The real part of the Reynolds stress represents the driving and damping terms for the GAM. The imaginary part of the Reynolds stress modifies the real frequency. In other words, the phase difference between the GAM oscillation and the nonlinear force by the Reynolds stress affects the real frequency.

3.1 Saturation amplitude

In general, the frequency is complex $\omega = \omega_r + i\gamma$, and γ determines the growth rate of the mode. When the nonlinear stabilization effects are included, γ correspondingly becomes smaller at an increase of the GAM amplitude. The growth of the GAM saturates when $\gamma = 0$. From now on, we omit the subscript of the real frequency as $\omega_r \to \omega$. The saturation amplitude of the GAM is determined by the real part of Eq. (10) with assumption of purely real ω as

$$u^2 = \frac{-\mathrm{Im}\varOmega(\omega) + \mathrm{Re}\varGamma_1(\omega)}{\mathrm{Re}\varGamma_2(\omega)}. \tag{11}$$

The first term in the numerator is the linear damping rate, the second term is the quasi-linear driving by the turbulence, which stems from the quasi-linear response of the Reynolds stress to u. The denominator reflects the nonlinear stabilization effect. Here, the GAM is assumed to satisfy the excitation condition $\mathrm{Im}\varOmega < \mathrm{Re}\varGamma_1$. We note that the frequency ω includes the nonlinear effects, and is a function of the amplitudes of the GAM and the turbulence. Therefore, the saturated amplitude is affected by the nonlinear frequency. The self-consistent solution can be obtained by solving Eq. (11) with the nonlinear frequency. In this article, the condition $u \ll 1$ is used, that is, the parameters are chosen near the threshold of the excitation condition.

3.2 Nonlinear frequency

The equation for the nonlinear frequency is obtained from the imaginary part of Eq. (10), which can be rewritten as

$$\varDelta \equiv \omega^2 - \omega_{\mathrm{L}}^2 + \left(\omega^2 - \frac{1}{2q^2}\right)\left\{\frac{c_1}{\varDelta\omega}\left(1 + \frac{\omega^2}{\varDelta\omega^2}\right)^{-1} - \frac{6c_2u^2}{\varDelta\omega}\left(1 + \frac{\omega^2}{\varDelta\omega^2}\right)^{-1}\left(1 + \frac{4\omega^2}{\varDelta\omega^2}\right)^{-1}\right\} = 0. \tag{12}$$

The first two terms produce the linear GAM frequency without the Reynolds stress. The third and fourth terms are the nonlinear effects, which result from the imaginary parts of the quasi-linear driving force and nonlinear damping force due to the ambient turbulence through the Reynolds stress. In other words, the origin of the nonlinear effects is the phase delay between the nonlinear force and the GAM oscillation.

The analytical expression for the nonlinear frequency is derived as follows. Recall that the ordering used here is $\varDelta\omega^{-2} \sim u^2 \ll 1$. We assume that $\varDelta\omega \sim u$ are less than unity, and that their squares are much less than unity. The dispersion relation can be expanded with this ordering in terms of $\varDelta\omega^{-2} \sim u^2 \ll 1$ as follows.

$$\varDelta = \varDelta^{(1)} + \varDelta^{(2)} + \cdots, \tag{13a}$$

$$\varDelta^{(1)} \equiv \omega^2 - \omega_{\mathrm{L}}^2 + \frac{c_1}{\varDelta\omega}\left(\omega^2 - \frac{1}{2q^2}\right), \tag{13b}$$

$$\varDelta^{(2)} \equiv -\left(\omega^2 - \frac{1}{2q^2}\right)\left\{\frac{c_1}{\varDelta\omega}\left(\frac{\omega}{\varDelta\omega}\right)^2 + 6\frac{c_2u^2}{\varDelta\omega}\right\}. \tag{13c}$$

The perturbative solution is assumed to have the form of $\omega = \omega_1 + \delta\omega$, where $\delta\omega/\omega_1 \ll 1$. The leading term ω_1 is the solution of $\varDelta^{(1)} = 0$, and the perturbation part $\delta\omega$ is evaluated from the relation

$$\delta\omega = -\varDelta^{(2)}(\omega_1)\left[\left.\frac{\partial\varDelta^{(1)}}{\partial\omega}\right|_{\omega=\omega_1}\right]^{-1}. \tag{14}$$

Then, the analytical solution is derived as

$$\omega_1^2 = \frac{\omega_{\mathrm{L}}^2 + c_1/(2q^2\varDelta\omega)}{1 + c_1/\varDelta\omega}, \tag{15a}$$

$$\delta\omega = \frac{1}{2\omega_1}\left(1 + \frac{c_1}{\varDelta\omega}\right)^{-2}\left\{\frac{c_1}{\varDelta\omega}\left(\frac{\omega_1}{\varDelta\omega}\right)^2 + 6\frac{c_2u^2}{\varDelta\omega}\right\}. \tag{15b}$$

The nonlinear frequency strongly depends on the nonlinear decorrelation rate $\varDelta\omega$ of the turbulence, which corresponds to the lifetime of the turbulent eddies. In the zero lifetime limit, the nonlinear effects disappear, and the solution approaches the solution in the linear case, ω_{L}. A finite lifetime of the turbulent eddies evidently leads to a frequency downshift, which can be estimated as

$$\omega_1 \sim \omega_{\mathrm{L}}\left(1 - \frac{c_1}{2\varDelta\omega}\right) \sim \omega_{\mathrm{L}}\left(1 - p^2\beta_{\mathrm{NL}}\right), \tag{16}$$

where β_{NL} is the order of $k_\theta^2/k_\perp^4 \sim k_\perp^{-2} \gg 1$. Therein we use the order estimate $\varDelta\omega \sim \omega_*$. The finite gyro-radius effect on the real frequency was investigated with linear gyro-kinetic theory, and the frequency shift is $\omega \approx \omega_{\mathrm{L}}(1 + \beta_{\mathrm{g}}p^2)$, where the coefficient β_{g} is of order unity [27–30,37], (i.e., much smaller than the shift from the nonlinear effect found in the present study). Thus the nonlinear frequency shift can greatly exceed the frequency shift due to the finite gyro-radius effects.

Numerical turbulence simulations show that the parameter β_{NL} can be as large as 40 [33], which is of similar magnitude as the result obtained in this study but with the opposite sign. However the model of [33] is different from our model as follows: a two-fluid model is employed (electron dynamics is included), the assumed turbulence is the ion temperature gradient mode with ballooning structures, the magnetic configuration includes the effect from the vicinity of the separatrix, and the dominant nonlinear effect is the turbulent transport modulation, which is not considered in this study. The set of normalized parameters in [33] is estimated as $k_\theta = 0.2$, $p = 0.1$, $c_1 = 0.3$, $c_2 = 0.2$, $\omega_* = 2$, $u = 2$. A normalized GAM amplitude of order unity is numerically obtained in [33], whereas we assume the small amplitude $u \ll 1$.

3.3 Nonlinear propagation

The characteristics of the radial propagation of the GAM are modified by the nonlinearity. The radial phase velocity v_{p} is defined as $v_{\mathrm{p}} = \omega/p$. The phase velocity becomes smaller due to the downshift of the GAM frequency induced by the turbulence effect. The radial group velocity v_{g} is

$$v_{\mathrm{g}} = \frac{\partial\omega}{\partial p} \approx \left(\frac{\partial c_1}{\partial p}\frac{\partial}{\partial c_1} + \frac{\partial c_2}{\partial p}\frac{\partial}{\partial c_2}\right)(\omega_1 + \delta\omega), \tag{17}$$

and can be written as

$$v_{\mathrm{g1}} \equiv \left(\frac{2c_1}{p}\frac{\partial}{\partial c_1} + \frac{2c_2}{p}\frac{\partial}{\partial c_2}\right)\omega_1 = -(\omega_1 p)^{-1}\frac{c_1}{\varDelta\omega}\left(1 + \frac{c_1}{\varDelta\omega}\right)^{-2}, \tag{18a}$$

$$\begin{aligned} v_{\mathrm{g2}} &\equiv \left(\frac{2c_1}{p}\frac{\partial}{\partial c_1} + \frac{2c_2}{p}\frac{\partial}{\partial c_2}\right)\delta\omega \\ &= (p\omega_1)^{-1}\left(1 + \frac{c_1}{\varDelta\omega}\right)^{-4}\left[\left(1 - \frac{c_1^2}{\varDelta\omega^2} + \frac{c_1}{2q^2\varDelta\omega}\right)\right. \\ &\quad \left.\times\left\{\frac{c_1}{\varDelta\omega}\left(\frac{\omega_1}{\varDelta\omega}\right)^2 + \frac{6c_2u^2}{\varDelta\omega}\right\} - \frac{c_1^2}{\omega_1^2\varDelta\omega^2}\left(\frac{\omega_1}{\varDelta\omega}\right)^2\right]. \end{aligned} \tag{18b}$$

Here we assume that only c_1 and c_2 have a p dependence. The dominant contribution to the group velocity is denoted as v_{g1}. The velocity v_{g2} is a smaller correction with $v_{\mathrm{g2}}/v_{\mathrm{g1}} \sim \varDelta\omega^{-2} \sim u^2 \ll 1$. Since $v_{\mathrm{g1}} < 0$, the sign of the group velocity is negative, which is the opposite of the linear prediction. When p is positive, the phase velocity is always positive (the phase of the GAM propagates outward), but the direction of the group velocity becomes negative due to the turbulence response. Therefore the phase and group velocities are in the opposite directions. The difference in the propagation directions of the phase and group velocities has also been observed in Landau-fluid ITG turbulence simulations [42]. In the present study, the group velocity is found to be strongly enhanced by the nonlinearities. The GAM frequency without the turbulence drive does not have a p-dependence in the present article so that the group velocity of the GAM without turbulence response becomes zero $v_g = \partial_p \omega = 0$. In this limit, the magnitude of the group velocity deviates from that of the phase velocity. Note again that linear dispersive properties of the GAM, which come from the finite Larmor radius effects, are neglected in this study. However, the turbulence Reynolds stress makes ω depend on p, and the magnitudes of the phase and the group velocities approach each other. The order of the ratio between the phase and group velocities is roughly $v_g/v_p \sim c_1/\Delta\omega \sim 1$, which is much larger than that predicted by the linear gyro-kinetic theory, $v_g/v_p \sim O(p^2) \sim 0.01$ [29, 30], where $p \sim 0.1$ is used. Thus the turbulence potentially causes a rapid radial propagation of the energy and momentum of the GAM.

4. Estimates Using Experimental Parameters

In this section, we discuss the validity of the assumptions in the nonlinear model, using typical experimental values. Next, we evaluate the nonlinear frequency and propagation velocity.

First, we discuss the consistency of the truncation model for the nonlinearity. The nonlinear Reynolds stress can be expanded in terms of the GAM vorticity u. The estimated experimental values of u are shown in Table 1 [5,6,21,43]. Because the normalized vorticity is estimated to be $u \sim 0.3$ in several experiments, the truncation of the Reynolds stress at the third order $O(u^3)$ is possible while maintaining ten percent accuracy.

Next, we evalute the nonlinear frequency shift of the GAM for the experiments at JFT-2M [21] and CHS [43]. The evaluated set of the normalized parameters (c_1, c_2/c_1, $1/\Delta\omega$, $c_1/\Delta\omega$, u) is shown in Table 2. By inserting the values from Table 1 into the analytical solution for the frequency Eqs. (15a), (15b), the nonlinear frequency shifts is $\omega/\omega_L \approx 0.88$ for the JFT-2M experiment and $\omega/\omega_L \approx 0.99$ for the CHS experiment. The frequency shift in the JFT-2M experiment is not negligible. The higher order corrections, $c_1\omega_1^2/\Delta\omega^3$, $c_2u^2/\Delta\omega$, have much smaller contributions, whose values are $c_1\omega_1^2/\Delta\omega^3 \sim 0.017$, $c_2u^2/\Delta\omega \sim 0.07$ for JFT-2M, and $c_1\omega_1^2/\Delta\omega^3 \sim 0.001$, $c_2u^2/\Delta\omega \sim 0.001$ for CHS. One clearly has to take the frequency shift of the GAM into account in the case of large amplitude turbulence. In this study, the effect of the Reynolds stress is focused on. It has to be noted that the dynamic shearing [9] is also predicted to be effective in the finite magnetic shear case as in the JFT-2M experiment.

The nonlinear propagation characteristic can be evaluated for the experiments at JFT-2M [21]. The ratio of the phase velocity to the group velocity is estimated to be $v_g/v_p \sim 0.4$ in this case, so the group velocity can be comparable to the phase velocity. Therefore, the nonlinear enhancement of the group velocity should be experimentally observable.

Table 1 Normalized vorticity (see Sec. 2.1) estimated from the experiments.

	JFT-2M [21]	CHS [43]	HL-2A [5]	DIIID [6]
u	0.3~0.7	0.08	0.3	0.3

Table 2 Set of normalized parameters (see Sec. 2.2) evaluated from the experiments.

	c_1	c_2/c_1	$1/\Delta\omega$	$c_1/\Delta\omega$	u
JFT-2M [21]	0.6	2.1	0.27	0.24	0.3
CHS [43]	0.01	1.6	0.33	0.01	0.075

5. Summary and Discussion

The nonlinear frequency shift and nonlinear propagation characteristics of GAMs driven by turbulence have been investigated. We have focused on plasmas with high safety factor and low magnetic shear. The nonlinear dispersion relation has been derived from the calculations of Reynolds stress based on the wave kinetic equation. Thereby we assumed the vorticity of the GAM to be smaller than the drift wave frequency, and have truncated the nonlinearity at the third order of the GAM amplitude.

The real frequency of the GAM is modified by the nonlinearities of the turbulence and the GAM; the turbulence effect introduces a frequency downshift, and the nonlinear GAM stabilization effect upshifts the frequency, with the net frequency shift being downward. The frequency shift stems from the quasi-linear driving force on the GAM which arises from the turbulence modulation. The magnitude of the frequency shift is much larger than expected from the linear theory. The frequency shift is a function of the square of the GAM wavenumber so that the radial group velocity is strongly affected by the nonlinearity. The propagation direction of the group velocity also changes as a result of the nonlinearity. When the radial wavenumber of the GAM is positive (outward radial phase velocity), the direction of the radial group velocity is inward. The radial group velocity is much larger than that found from the linear theory, and can be comparable to the radial phase velocity. The Reynolds stress does not only serve as a driving force of the GAM, but also modifies its characteristics by leading to a nonlinear dispersion relation.

Finally, by using realistic experimental parameters, we discuss the limitations of our simple analytical model.

It is to be noted that the bicoherence analysis in the JFT-2M experiment indicates the importance of the nonlinear coupling of the density fluctuations of the GAMs [44]. Moreover, it has been reported that the dynamical shearing of the turbulence by the GAMs works as a driving force for the GAMs for the large magnetic shear [9]. A unified study of the nonlinearities of density fluctuation and the Reynolds stress on the dispersion relation is left to the future. In addition, in some scenarios, it may be invalid to truncate the nonlinearity at the third order of the normalized GAM vorticity for $u \sim 1$ [21]. Thus, an analytical computation of the Reynolds stress for $u \sim 1$ is left to future work.

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