Canonical Hamiltonian Model of Reduced MHD and its Comparison with the Two-Dimensional Euler System

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While expressing the ideal fluid/plasma equations in terms of Eulerian variables, we encounter a non-canonical Hamiltonian structure. In other words, Poisson operators determining symplectic geometry have non-trivial kernels that foliate phase spaces. There are several different recipes for "canonicalizing" such Hamiltonian formalisms by either reducing or extending phase spaces. Clebsch parametrization is a well-known method for reducing phase spaces. Here we introduce a new scheme that generalizes the Clebsch parametrization. Using the new set of variables, we delineate a fundamental difference between the reduced magnetohydrodynamic equations and the two-dimensional Euler equations.

Keywords: reduced MHD, Euler equation, Hamiltonian formalism, Clebsch parametrization, canonicalization

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1. Introduction

Reduced magnetohydrodynamic (RMHD) equations are used to describe the dynamics of plasma in the limit of low-β [1–4], which may be viewed as an extension of the two-dimensional Euler vorticity (2DEV) equation for an ideal incompressible fluid by coupling with an electromagnetic field. One may express the RMHD or 2DEV equation in a non-canonical Hamiltonian form [5, 6], which has certain conservation laws arising from both symmetries in the Hamiltonian and a "topological defect" (kernel) of the Poisson bracket. The constants of motion associated with the latter, which are typical in a non-canonical system, are called Casimir invariants. A Casimir invariant foliates the phase space, so that the dynamics is constrained on a leaf of the Casimir invariant.

Clebsch parametrization [7] of Eulerian variables is often a useful method of canonicalizing a non-canonical system [8, 9]. The key is to express the vorticity in a Clebsch 2-form [10]. The vorticity equation of 2DEV is then divided into a pair of equations governing the canonical variables. There is a fundamental relationship between the Clebsch parameters and the Lagrangian description of fluid motion [11]; the latter is naturally canonical as an adjoint representation of the Lie algebra of diffeomorphism.

Interestingly, in the RMHD equations, a canonical variable may be chosen to be an arbitrary function of the Clebsch parameter of the magnetic field (the magnetic flux function), and the equations of motion can be expressed in a generalized flux coordinate representation. In this study, we show that different canonical representations have different nonlinear terms, each describing a "sub-class" of nonlinear dynamics with a different "strength" of nonlinearity.

2. Hamiltonian Model

2.1 Euler fluid

We begin with a short review of the Hamiltonian formalism. Let u be a state vector belonging to a Hilbert space V and $H(u)$ be a Hamiltonian (a smooth functional on V). A general Hamilton's equation of motion may be expressed as

$$\partial_t u = \mathcal{J}\partial_u H(u),$$

where $\mathcal{J}$ is an antisymmetric operator called the *Poisson operator* [11]. Motion occurs in the direction perpendicular to the gradient of $H(u)$ (hence, in orbits). Thus, the energy (Hamiltonian) is conserved.

In a canonical Hamiltonian system, the Poisson operator can be represented by a symplectic operator

$$\mathcal{J}_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

in an appropriate coordinate system. A more general Hamiltonian system may have a general Poisson operator that cannot be transformed into the form of J_c [12]. In particular, when J has a nontrivial kernel (thus, cokernel), we say that the system is "non-canonical". A functional $C(u)$ such that $\partial_u C(u) \in \mathrm{Ker}(J)$ is called a Casimir invariant.

Now, we formulate the Hamiltonian and Poisson operators of the 2DEV equation. Let U be the vorticity of an ideal two-dimensional incompressible fluid, which obeys 2DEV (the *curl* of Euler's equation):

$$\partial_t U + [\phi, U] = 0, \tag{1}$$

where the bracket is defined as

$$[f, g] = \partial_x f \partial_y g - \partial_y f \partial_x g, \tag{2}$$

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and ϕ is the stream function ($\Delta\phi = U$). The fluid energy can be expressed as

$$H = \frac{1}{2}\int |\nabla\phi|^2 \mathrm{d}^2x = -\frac{1}{2}\int \Delta^{-1}U \cdot U \mathrm{d}^2x, \quad (3)$$

where Δ^{-1} is the inverse operator of Δ. Using Eq. (3) as a Hamiltonian represented on a function space of U, we may express Eq. (1) in a Hamiltonian form:

$$\partial_t U = \mathcal{J}(U)\partial_U H(U), \quad (4)$$

where $\mathcal{J}(U)\circ = -[U, \circ]$ ($\circ$ means insertion). The corresponding Poisson bracket is

$$\begin{aligned}\{F, G\} &= \langle \partial_U F, \mathcal{J}(U)\partial_U G\rangle \\ &= -\int \partial_U F[U, \partial_U G]\mathrm{d}^2x \\ &= \int U[\partial_U F, \partial_U G]\mathrm{d}^2x,\end{aligned}$$

which is a typical Lie-Poisson bracket (a Lie-Poisson bracket is known to satisfy Jacobi's identity [9]). Evidently, for an arbitrary smooth function f, a functional

$$C(U) = \int f(U)\mathrm{d}^2x, \quad (5)$$

is a Casimir invariant; i.e., $\{C, F\} = 0$ for every $F(U)$. The enstrophy $\int U^2\mathrm{d}^2x$ is a special case of Eq. (5).

Introducing Clebsch parameters Q and P, we can formulate a "canonicalized" sub-class of the 2DEV in Eq. (4). Let us set

$$U = [Q, P], \quad (6)$$

which may be viewed as a Clebsch 2-form [10]. In terms of Q and P, the Hamiltonian of Eq. (3) is rewritten as

$$H(Q, P) = -\frac{1}{2}\int \Delta^{-1}[Q, P] \cdot [Q, P]\mathrm{d}^2x. \quad (7)$$

Invoking the canonical Poisson operator (symplectic operator)

$$\mathcal{J}_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

we consider the canonical Hamilton's equation

$$\begin{aligned}\partial_t \begin{pmatrix} Q \\ P \end{pmatrix} &= \mathcal{J}_C \begin{pmatrix} \partial_Q H \\ \partial_P H \end{pmatrix} = \mathcal{J}_C \begin{pmatrix} [\phi, P] \\ -[\phi, Q] \end{pmatrix} \\ &= \begin{pmatrix} -[\phi, Q] \\ -[\phi, P] \end{pmatrix}.\end{aligned} \quad (8)$$

These equations are consistent with the original Eq. (4). In fact,

$$\begin{aligned}\partial_t U &= [\partial_t Q, P] + [Q, \partial_t P] \\ &= [P, [\phi, Q]] - [Q, [\phi, P]] \\ &= -[\phi, [Q, P]] = -[\phi, U],\end{aligned}$$

where we used Jacobi's identity

$$[f, [g, h]] + [g, [h, f]] + [h, [f, g]] = 0. \quad (9)$$

However, the system of Eq. (8) is not equivalent to Eq. (4) because the Clebsch 2-forms in Eq. (6) are not "complete", i.e., they do not span the entire space of 2-forms [10]. In fact, U of the form in Eq. (8) is restricted to have zero circulation:

$$\int U\mathrm{d}^2x = \int [Q, P]\mathrm{d}^2x = 0. \quad (10)$$

Similarly, we observe

$$\begin{aligned}C_{ch} &= \int QU\mathrm{d}^2x \\ &= \int Q[Q, P]\mathrm{d}^2x = 0.\end{aligned} \quad (11)$$

This constant of motion corresponds to the "cross-helicity", which will be given an important role in RMHD (see also [13]).

By these constraints, the system of Eq. (8) becomes a sub-class of the original 2DEV equation, Eq. (4), whereas the number of variables increases. As U obeys Eq. (4), both Q and P are transported by the same flow $[\phi, \circ]$. However, the excess variable Q will be given an important role in RMHD.

2.2 Reduced MHD

The RMHD equations can be applied to the analysis of a large-aspect-ratio tokamak (Fig. 1). The ideal MHD equations are expressed as

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \quad (12)$$

$$\partial_t \boldsymbol{U} + \nabla\times(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = \nabla\times(\boldsymbol{B}\cdot\nabla)\boldsymbol{B}, \quad (13)$$

$$\nabla\cdot\boldsymbol{B} = 0, \quad \nabla\cdot\boldsymbol{v} = 0, \quad (14)$$

for a magnetic field $\boldsymbol{B}(\boldsymbol{x}, t)$ and a vorticity field $\boldsymbol{U}(\boldsymbol{x}, t)$. Further, $\boldsymbol{v}(\boldsymbol{x}, t)$ is the fluid velocity. We consider the following ordering in terms of the inverse aspect ratio, $\epsilon = a/R_0$, where R_0 is the toroidal radius, and a is the poloidal radius:

$$\begin{aligned}&\partial_x, \partial_y \sim 1, \;\; \partial_z \sim \epsilon, \;\; \partial_t \sim \epsilon, \\ &B_x, B_y \sim \epsilon, \;\; B_z \sim 1 + \epsilon^2, \;\; v_x, v_y \sim \epsilon.\end{aligned} \quad (15)$$

We introduce the poloidal flux ψ that is connected to a magnetic field:

$$\boldsymbol{B} = B_z\boldsymbol{e}_z + \boldsymbol{e}_z \times \nabla\psi. \quad (16)$$

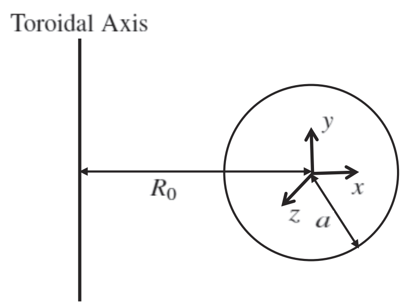

Fig. 1 Geometry of the tokamak coordinate system.

The equations of ϵ^2 ordering yield an RMHD system,

$$\partial_t \psi + [\phi, \psi] = 0, \tag{17}$$

$$\partial_t U + [\phi, U] = [\psi, J], \tag{18}$$

where the current density is denoted by $J = \Delta\psi$.

Defining the Hamiltonian and Poisson operator as

$$H = \frac{1}{2}\int \left(|\nabla\phi|^2 + |\nabla\psi|^2\right) \mathrm{d}^2 x$$
$$= -\frac{1}{2}\int \left(\Delta^{-1}U \cdot U + \Delta\psi \cdot \psi\right) \mathrm{d}^2 x, \tag{19}$$

$$\mathcal{J}(U,\psi) = \begin{pmatrix} -[U,\circ] & -[\psi,\circ] \\ -[\psi,\circ] & 0 \end{pmatrix}, \tag{20}$$

respectively, we may express the RMHD equations in a non-canonical Hamiltonian form [6]:

$$\partial_t \begin{pmatrix} U \\ \psi \end{pmatrix} = \mathcal{J}(U,\psi) \begin{pmatrix} \partial_U H \\ \partial_\psi H \end{pmatrix}. \tag{21}$$

The corresponding Poisson bracket is

$$\{F,G\} = \int W_{ij}\left[\partial_{\xi_i}F, \partial_{\xi_j}G\right] \mathrm{d}^2 x,$$
$$W_{ij} = \begin{bmatrix} 0 & \psi \\ \psi & U \end{bmatrix},$$

where $(\xi_1, \xi_2) = (\psi, U)$. Now, we canonicalize Eq. (21). In the Clebsch parametrization in Eq. (6), we relate Q, P with ψ; if we set $\psi = \psi(Q, P)$, the canonical equations of motion become

$$\partial_t \begin{pmatrix} Q \\ P \end{pmatrix} = \mathcal{J}_C \begin{pmatrix} \partial_Q H \\ \partial_P H \end{pmatrix}$$
$$= \begin{pmatrix} -[\phi, Q] - (\partial_P\psi)J \\ -[\phi, P] + (\partial_Q\psi)J \end{pmatrix}. \tag{22}$$

In comparison with Eq. (8), the RMHD system in Eq. (22) includes the additional terms representing the magnetic force.

Here, $\psi(Q, P)$ represents the following two types of settings. If we set $\psi = Q^\alpha P^\beta$ (α and β are real constants), the canonical equations become

$$\partial_t \begin{pmatrix} Q \\ P \end{pmatrix} = \mathcal{J}_C \begin{pmatrix} \partial_Q H \\ \partial_P H \end{pmatrix}$$
$$= \begin{pmatrix} -[\phi, Q] - \beta Q^\alpha P^{\beta-1} J \\ -[\phi, P] + \alpha Q^{\alpha-1} P^\beta J \end{pmatrix}. \tag{23}$$

Alternatively, setting $\psi = Q^\alpha + P^\beta$, we obtain

$$\partial_t \begin{pmatrix} Q \\ P \end{pmatrix} = \mathcal{J}_C \begin{pmatrix} \partial_Q H \\ \partial_P H \end{pmatrix}$$
$$= \begin{pmatrix} -[\phi, Q] - \beta P^{\beta-1} J \\ -[\phi, P] + \alpha Q^{\alpha-1} J \end{pmatrix}. \tag{24}$$

Interestingly, the magnetic term (including the current J) changes depending on the representation of the magnetic flux function. Setting $J = 0$ decouples Q and P, and then the systems of Eqs. (23) and (24) reduce to 2DEV in Eq. (8).

In Table 1, we summarize various forms of the 2DEV and RMHD equations. The corresponding Hamiltonians and Poisson operators are given in Table 2.

As in 2DEV, the canonicalized systems are subclasses of the general non-canonical RMHD system in Eq. (21). Let us examine the relationships between the original non-canonical RMHD system and its canonicalized systems by observing the Casimir invariants,

$$C(U,\psi) = \int U h(\psi) \mathrm{d}^2 x, \tag{25}$$

where h is an arbitrary smooth function. The cross-helicity [10] is a special case of C with $h(\psi) = \psi$. In the canonicalized system of $\psi = Q^\alpha$, the cross-helicity is fixed at zero;

$$C_{ch} = \int Q^\alpha [Q, P] \mathrm{d}^2 x$$
$$= \frac{1}{\alpha+1}\int [Q^{\alpha+1}, P] \mathrm{d}^2 x = 0, \tag{26}$$

where C_{ch} is the cross-helicity. Similarly, the systems in Eqs. (23) and (24) also assume zero cross-helicity. An advantage of the formulations of Eq. (22), in comparison to the formula given in [6], is that we can solve equations with fewer variables. In numerical analysis, this is an advantage. Then, because the interactions that contain the current density are represented by the power of canonical variables, local algebraic analysis is possible (Section 3). We must note that formulations containing higher powers are difficult to solve.

3. Application to Local Analysis

As an application of the canonicalized Hamiltonian formalism, we study the effect of the magnetic term in an RMHD system using a local analysis. Setting $\alpha = 2, \beta = 1$ in Eq. (23), we obtain

$$\partial_t Q + [\phi, Q] = -Q^2 J, \tag{27}$$

$$\partial_t P + [\phi, P] = 2QPJ. \tag{28}$$

The poloidal flux is $\psi = Q^2 P$ and the current density is $J = \Delta(Q^2 P)$. We study the time evolution of the current density near the X-point of the magnetic field by "local analysis" (in [14], Imshennik and Syrovatskii analyzed the compressible fluid model by local analysis; see also Biskamp [15] for a local analysis of incompressible MHD).

We consider a local solution of Eqs. (27) and (28), such as

$$Q = a(t) + b(t)y, \tag{29}$$

$$P = c(t)x + d(t)y. \tag{30}$$

Table 1 The Equations of motion and their canonicalized sub-classes.

	General non-canonical systems	Canonicalized sub-classes
Euler	$\partial_t U + [\Delta^{-1}U, U] = 0$	$U = [Q, P]$
		$\partial_t Q + [\Delta^{-1}[Q, P], Q] = 0$ $\partial_t P + [\Delta^{-1}[Q, P], P] = 0$
RMHD	$\partial_t \psi + [\Delta^{-1}U, \psi] = 0$ $\partial_t U + [\Delta^{-1}U, U] = [\psi, \Delta\psi]$	$U = [Q, P],\ \psi = \psi(Q, P)$
		$\partial_t Q + [\Delta^{-1}[Q, P], Q] = -(\partial_P \psi)J$ $\partial_t P + [\Delta^{-1}[Q, P], P] = (\partial_Q \psi)J$ $J = \Delta\psi(Q, P)$

Table 2 Hamiltonians and Poisson operators for several models.

	Hamiltonians	Poisson operators
Euler	Non-canonical	$\mathcal{J}(U) = -[U, \circ]$
	$H(U) = -\frac{1}{2}\int \Delta^{-1}U \cdot U \mathrm{d}^2x$	
	Canonical: $U = [Q, P]$	$\mathcal{J}_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
	$H(Q, P) = -\frac{1}{2}\int \Delta^{-1}[Q, P] \cdot [Q, P]\mathrm{d}^2x$	
RMHD	Non-canonical	$\mathcal{J}(U, \psi) = \begin{pmatrix} -[U, \circ] & -[\psi, \circ] \\ -[\psi, \circ] & 0 \end{pmatrix}$
	$H(U, \psi) = -\frac{1}{2}\int \left(\Delta^{-1}U \cdot U + \Delta\psi \cdot \psi\right) \mathrm{d}^2x$	
	Canonical: $U = [Q, P],\ \psi = \psi(Q, P)$	$\mathcal{J}_C = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$
	$H(Q, P) = -\frac{1}{2}\int \left(\Delta^{-1}[Q, P] \cdot [Q, P] + \Delta\psi(Q, P) \cdot \psi(Q, P)\right) \mathrm{d}^2x$	

Then, the physical quantities are given by

$$\psi = Q^2 P = (a + by)^2(cx + dy), \tag{31}$$

$$J = \Delta\psi = \Delta(Q^2 P) = 2b^2(cx + dy) + 4bd(a + by), \tag{32}$$

$$U = \Delta\phi = [Q, P] = -bc. \tag{33}$$

Here we solve Eq. (33) as $\phi = -(bc/2)y^2$. In comparison with the previous model of [15], the present model includes a finite magnetic force $[\psi, J]$, which yields an essential difference between RMHD and 2DEV. Substituting Eqs. (29) and (30) into Eqs. (27) and (28), and comparing the zero- and first-order terms, we derive the ordinary differential equations

$$\dot{a} = -4a^3 bd, \tag{34}$$

$$\dot{b} = -14a^2 b^2 d, \tag{35}$$

$$\dot{c} = 8a^2 bcd, \tag{36}$$

$$\dot{d} = 8a^2 bd^2, \tag{37}$$

where the dot denotes the time derivative. Here we assumed that the term including x is negligibly small in Eq. (27); this assumption limits the range of x in the latter calculation. We also assumed that the terms $[\phi, Q]$ and $[\phi, P]$ are negligibly small; the former turns out to be zero, and the latter to be slowly increasing. Integrating Eqs. (34)–(37), we obtain

$$a \propto (t - t_0)^{-2/7}, \tag{38}$$

$$b \propto (t - t_0)^{-1}, \tag{39}$$

$$c \propto (t - t_0)^{4/7}, \tag{40}$$

$$d \propto (t - t_0)^{4/7}, \tag{41}$$

where t_0 is a constant. For $t_0 > 0$, a and b blow up as $t \to t_0$.

The finite-time blow-up of these coefficients causes a singularity in the current density J, which consists of the terms

$$4abd \propto (t - t_0)^{-\frac{2}{7}-1+\frac{4}{7}} = (t - t_0)^{-\frac{5}{7}},$$

$$2b^2cx \propto (t - t_0)^{-2+\frac{4}{7}}x = (t - t_0)^{-\frac{10}{7}}x,$$

$$6b^2dy \propto (t - t_0)^{-2+\frac{4}{7}}y = (t - t_0)^{-\frac{10}{7}}y.$$

Now we examine the consistency of the assumptions made in the above derivation. First, we evaluate the convective terms. Obviously, $[\phi, Q] = 0$, whereas $[\phi, P]$ behaves as

$$bc^2y \propto (t - t_0)^{-1+\frac{8}{7}}y = (t - t_0)^{\frac{1}{7}}y.$$

Thus, it is smaller than the other exploding terms. The first-order terms on the right-hand side of Eq. (27) scale as

$$a^2b^2cx \propto (t-t_0)^{-\frac{4}{7}-2+\frac{4}{7}}x = (t-t_0)^{-2}x,$$
$$a^2b^2dy \propto (t-t_0)^{-\frac{4}{7}-2+\frac{4}{7}}y = (t-t_0)^{-2}y.$$

The term including x has the same singularity. To omit this term, we have to consider a domain such that $|x| \ll |y|$.

4. Concluding Remarks

In this study, we constructed a series of canonical Hamiltonian systems that describe self-consistent subclasses of RMHD. They have a common symplectic manifold that has a different relationship to the Poisson manifold of the original non-canonical Hamiltonian system (general RMHD). Omitting the magnetic term in the Hamiltonian of each system, the systems reduces to 2DEV. In contrast, the phase space of 2DEV can be extended by combining a passive variable ψ that is transported by the same potential $\phi = \Delta^{-1}U$ (in RMHD, this ψ is not an artificial passive quantity, but the magnetic flux that influences the dynamics of U through the magnetic term), i.e.,

$$\partial_t U + [\phi, U] = 0, \tag{42}$$
$$\partial_t \psi + [\phi, \psi] = 0, \tag{43}$$

where $\phi = \Delta^{-1}U$. This trivially extended system is non-canonical. One may assume that $P = \psi$ embeds the symplectic manifold (P, Q) of Eq. (8) into the space (U, ψ) as $(U, \psi) = ([Q, P], P)$, which is considerably complex. As shown in Eq. (10), this embedding (or symplectic foliation) is constrained by $\int U \mathrm{d}^2x = 0$.

One may consider several applications of the present formulation. For example, the conservation of the Casimir invariants is automatic; so one may formulate a numerical scheme that guarantees the constancy of the Casimir invariants. Another interesting application can be made in the study of a singularity (finite-time blow-up); as shown in Table 1, we may formulate a series of nonlinear systems with different degrees of nonlinearity representing the magnetic force, each of which is a sub-class of the dynamics of the original system with a different strength of the nonlinear effect. In the present study, we have constructed a “local solution” that has a singularity. However, in a finite-sized domain, the global solution may exhibit different behaviors. Numerical simulations based on the present formulation will be reported elsewhere.

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