Fluid Moments in the Reduced Model for Plasmas with Large Flow Velocity

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Representation of particle fluid moments in terms of fluid moments in the modified guiding-centre model for flowing plasmas with large $\boldsymbol{E} \times \boldsymbol{B}$ velocity [N. Miyato *et al.*, J. Phys. Soc. Jpn. **78**, 104501 (2009)] is derived from the formal exact representation by a perturbative expansion in the subsonic flow case. It is similar to that in the standard gyrokinetic model in the long wavelength limit, except it has an additional flow term. The flow term has no effect on the representation for particle density, leading to the same representation as the standard one formally. In the conventional guiding-centre models for flowing plasmas, on the other hand, the representation for particle density is different from the standard one. This is due to the difference in the transformation for the guiding-centre position. Although the exact representation usually used in the standard gyrokinetic model has a different form from that in the modified guiding-centre case, correspondence between the two models is shown by considering the alternative form of exact representation in the standard gyrokinetic case. The representation for particle density is also obtained from the single particle Lagrangian by a variational method which is used to derive the representation in the transonic case.

Keywords: push-forward representation, guiding-centre transformation, flowing plasma, transport barrier

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1. Introduction

Reduced models such as the gyrokinetic model [1] in which fast gyro-motion of charged particles is separated from the other dynamics are widely used to study phenomena whose timescales are much longer than the period of the gyro-motion. Although a large plasma flow or a strong radial electric field is observed in transport barriers [2, 3] whose formation is necessary for confinement improvement of magnetised fusion plasmas, the large flow is not taken into account in the standard reduced model. Hence, extension of the standard model is needed to treat the formation of transport barriers. Several gyrokinetic Lagrangians have been presented to treat this situation [4–10]. Here we give an account of the underlying representations of the particle density in each of these models. We emphasise the similarity of the physics despite these differences which pertain to representation, not content. Ultimately, the basis on a system Lagrangian affords the ability to establish the underlying representation in a straightforward manner, via application of the functional derivative with respect to the electrostatic potential. This can be referred to as the variational method of the push-forward transformation.

Recently a reduced fundamental 1-form (phase space Lagrangian) including a large $\boldsymbol{E} \times \boldsymbol{B}$ flow was derived by modifying the standard guiding-centre transformation through the Lie-transform perturbation method [4]. In contrast to conventional models for a flowing plasma [5–10], the symplectic part of the Lagrangian is the same as the standard one without flow formally. In the reduced models the separation of the fast gyro-motion from the other dynamics is achieved by the phase space transformation from the particle phase space to the guiding-centre phase space.

Guiding-centre fluid moments defined as integrals in the guiding-centre phase space are different from corresponding particle fluid moments due to finite-Larmor-radius (FLR) effects [1, 11–21]. Any particle fluid moment can be represented in terms of the guiding-centre fluid moments. It is called the push-forward representation of particle fluid moment associated with the transformation from the particle phase space to the guiding-centre phase space [1, 20]. An example of the representation is found in the Poisson equation for an electrostatic potential or the quasi-neutrality condition in the gyrokinetic and gyrofluid models [1, 11–14, 20, 22]. The push-forward representation of particle density appears at the charge density part of the reduced Poisson equation which is written by an integral in the guiding-centre phase space with the guiding-centre distribution function. The requirement that the charge density part agrees with the product of particle density multiplied by electric charge for each particle species gives the push-forward representation of particle density. In the reduced models, the quasi-neutrality condition between electrons and singly charged ions shows the same relation since

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the difference between the particle density and the guiding-centre density can be neglected for electrons, while it is not for hot ions in the magnetised fusion plasmas. The difference between the particle density and the guiding-centre density can be viewed as polarisation density. The polarisation density also appears as a vorticity in traditional reduced fluid models which are not derived from reduced kinetic equations and therefore are not described by the guiding-centre fluid moments [23–26].

Reduced fluid equations are traditionally derived from more rigorous fluid models like the Braginskii model. In the derivations, it is needed to calculate the stress tensor explicitly and one encounters the issue of gyroviscous cancellations and corrections whose calculation is very cumbersome [27–29]. There is a detour to avoid the complicated issue on the stress tensor. The reduced fluid equations can be also derived from fluid moment equations of the reduced kinetic equation by using the push-forward representation [13, 16, 18]. In this alternative way, it is no more necessary to handle the stress tensor, since dynamical reduction is performed at the kinetic level. Besides, expressions for the gyroviscous force are obtained by comparing the FLR-corrected reduced fluid equations obtained from the guiding-centre fluid equations with the particle fluid momentum equation [13, 16].

Explicit push-forward representation depends on details of the guiding-centre transformation. In this paper we derive the push-forward representation of particle fluid moment in our modified guiding-centre model for flowing plasmas and compare it with those in the conventional guiding-centre models for flowing plasmas and in the standard gyrokinetic model with slow flow. In Sec. 2, we explain the modified guiding-centre model briefly. The stress is placed on the difference between the modified model and the conventional models. In Sec. 3, the push-forward representation of general scalar particle fluid moment is derived from the exact representation perturbatively in the subsonic flow case and correspondence to the standard gyrokinetic case is discussed. Pull-back representation of guiding-centre fluid moments, inverse of the push-forward representation, is derived in Sec. 4. In Sec. 5, we explain variational derivation of the push-forward representation of particle density by which the representation in the transonic case is derived. In Sec. 6, the push-forward representation of particle flux is considered. Finally, a summary is given in Sec. 7.

2. Guiding-Centre Theory

We consider a transformation from particle coordinates $z = (\boldsymbol{x}, v_\parallel, w, \theta)$ to guiding-centre coordinates $\boldsymbol{Z} = (\boldsymbol{X}, U, \mu, \xi)$ given by [4]

$$\boldsymbol{X} = \boldsymbol{x} - \epsilon \boldsymbol{\rho}_{\rm gc} + O(\epsilon^2), \tag{1}$$

$$U = v_\parallel + \epsilon G_1^U + O(\epsilon^2), \tag{2}$$

$$\mu = \frac{m w^2}{2B} + \epsilon G_1^\mu + O(\epsilon^2), \tag{3}$$

$$\xi = \theta + \epsilon G_1^\xi + O(\epsilon^2), \tag{4}$$

where $\boldsymbol{x}$ is the position of a particle with mass m and electric charge q, $v_\parallel$ is the particle velocity along the magnetic field, $w = |\boldsymbol{w}| = |\boldsymbol{v}_\perp - \boldsymbol{D}|$ is the particle perpendicular velocity in the frame moving with the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity $\boldsymbol{D}$, θ is the particle gyrophase angle, and $\epsilon \sim \rho/L$ is the small parameter with the Larmor radius ρ and the background gradient scale length L. G_1^i is a component of the vector field generating the guiding-centre transformation at first order in ϵ and is summarised in Appendix A. The perpendicular velocity vector is expressed by $\boldsymbol{w} = w\hat{c}$ with

$$\hat{c} = -\sin\theta \boldsymbol{e}_1 - \cos\theta \boldsymbol{e}_2, \tag{5}$$

where $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ are unit vectors spanning the plane perpendicular to $\hat{b} \equiv \boldsymbol{B}/B$. It is noted that the Larmor radius vector $\boldsymbol{\rho}_{\rm gc} = \hat{b}\times\boldsymbol{v}_\perp/\Omega$ has a non-vanishing gyroaverage part due to the definition of the gyrophase angle here:

$$\langle \boldsymbol{\rho}_{\rm gc} \rangle = \frac{\hat{b}\times\boldsymbol{D}}{\Omega} \equiv \boldsymbol{\rho}_E, \tag{6}$$

where $\Omega = qB/m$ is the gyrofrequency and $\langle\cdot\rangle$ denotes gyroaverage. The gyrophase dependent part of $\boldsymbol{\rho}_{\rm gc}$ is given by

$$\tilde{\boldsymbol{\rho}}_{\rm gc} = \boldsymbol{\rho}_{\rm gc} - \langle \boldsymbol{\rho}_{\rm gc} \rangle = \frac{\hat{b}\times\boldsymbol{w}}{\Omega} \equiv \boldsymbol{\rho}. \tag{7}$$

When it is assumed that the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity is comparable to the ion thermal velocity v_{ti}, $\boldsymbol{\rho}_E \sim \boldsymbol{\rho}$ for ions. The difference from the guiding-centre transformation without flow, $\boldsymbol{\rho}_E$, corresponds to the gyroaverage of the gyro-centre displacement vector in the standard gyrokinetics [20]. It is noted that G_1^μ also has the nonvanishing gyroaveraged part due to the $\boldsymbol{E}\times\boldsymbol{B}$ flow [4],

$$\langle G_1^\mu \rangle \simeq -\frac{\mu}{\Omega}\hat{b}\cdot\nabla\times\boldsymbol{D} \simeq -\nabla\cdot\left(\frac{\mu}{\Omega B}\nabla_\perp\varphi\right), \tag{8}$$

where φ is the electrostatic potential. The guiding-centre transformation above yields the following guiding-centre phase space Lagrangian,

$$\Gamma = L_p {\rm d}t = q\boldsymbol{A}^*\cdot{\rm d}\boldsymbol{X} + \epsilon^2\frac{m}{q}\mu{\rm d}\xi - H{\rm d}t, \tag{9}$$

where $\boldsymbol{A}^* = \boldsymbol{A} + \epsilon(m/q)U\hat{b}$ is the modified vector potential and H is the guiding-centre Hamiltonian. We assume that a magnetic field is independent of time. As mentioned above, the symplectic part of the Lagrangian is the same as the no flow case formally. In the conventional guiding-centre models for flowing plasmas, $\boldsymbol{A}^*$ includes a flow term which changes the symplectic structure in the no flow case when the flow is time-varying [5–9]. The phase space Lagrangian similar to the present one is also found in Ref. [19]. The Jacobian of the transformation from the particle phase space to the guiding-centre phase space is given by $\mathcal{J} = B_\parallel^*/m$ with $B_\parallel^* \equiv \hat{b}\cdot\boldsymbol{B}^*$ and $\boldsymbol{B}^* \equiv \nabla\times\boldsymbol{A}^*$. In our model the Jacobian is time-independent even in the time-varying

Table 1 Comparison among the no flow model, the Hamiltonian and Symplectic models for flowing plasmas.

	No flow	Hamiltonian	Symplectic
$\boldsymbol{\rho}_{\rm gc}$	$\boldsymbol{\rho}$	$\boldsymbol{\rho}+\boldsymbol{\rho}_E$	$\boldsymbol{\rho}$
$\boldsymbol{A}^*$	$\boldsymbol{A}+(m/q)U\hat{b}$	$\leftarrow$	$\boldsymbol{A}+(m/q)(\boldsymbol{D}+U\hat{b})$
H	$q\varphi+\frac{m}{2}U^2+\mu B$	$q\varphi+\frac{m}{2}U^2+\mu B-\frac{m}{2}D^2$	$q\varphi+\frac{m}{2}U^2+\mu B+\frac{m}{2}D^2$
$B_\parallel^*$	$B+(m/e)U\hat{b}\cdot\nabla\times\hat{b}$	$\leftarrow$	$B+(m/e)\hat{b}\cdot\nabla\times(\boldsymbol{D}+U\hat{b})$
p_ζ	$qA_\zeta+mUb_\zeta$	$\leftarrow$	$qA_\zeta+mD_\zeta+mUb_\zeta$

flow case, while it can have time dependence through the flow term in $\boldsymbol{A}^*$ in the conventional models. Moreover, since the symplectic part of the Lagrangian is the same as the one without flow, the guiding-centre Hamilton equations, $\dot{Z}^i=\{Z^i,H\}$, keep the general form in the no flow case:

$$\dot{\boldsymbol{X}}=\epsilon^{-1}\frac{\boldsymbol{B}^*}{mB_\parallel^*}\frac{\partial H}{\partial U}+\frac{\hat{b}}{qB_\parallel^*}\times\nabla H, \tag{10}$$

$$\dot{U}=-\epsilon^{-1}\frac{\boldsymbol{B}^*}{mB_\parallel^*}\cdot\nabla H, \tag{11}$$

$$\dot{\mu}\equiv 0, \tag{12}$$

$$\dot{\xi}=\epsilon^{-2}\frac{q}{m}\frac{\partial H}{\partial\mu}. \tag{13}$$

On the other hand, additional terms with time derivative of $\boldsymbol{D}$ appear in the conventional models and the Hamilton equation for the guiding-centre position is written as

$$\dot{\boldsymbol{X}}=\epsilon^{-1}\frac{\boldsymbol{B}^*}{mB_\parallel^*}\frac{\partial H}{\partial U}+\frac{\hat{b}}{qB_\parallel^*}\times\nabla H+\epsilon\frac{m}{qB_\parallel^*}\hat{b}\times\frac{\partial\boldsymbol{D}}{\partial t}. \tag{14}$$

The additional term in the above shows the polarisation drift.

Another advantage of the model is the expression of the toroidal angular momentum. If the Lagrangian does not depend on the toroidal angle ζ, the Euler-Lagrange equation or Noether's theorem states that its canonically conjugate momentum

$$p_\zeta\equiv\frac{\partial L_p}{\partial\dot{\zeta}}=eA_\zeta^*=eA_\zeta+mUb_\zeta \tag{15}$$

is a constant of motion, where A_ζ and b_ζ are covariant ζ components of $\boldsymbol{A}$ and $\hat{b}$, respectively. The toroidal angular momentum p_ζ is the same as the one without flow since the symplectic part of the Lagrangian is common. Therefore, we can follow the standard analysis for guiding-centre orbits [30]. Comparison results among the no flow model, the present model and the conventional models for flowing plasmas are summarised in Table 1 in which the guiding-centre Hamiltonian up to $O(\epsilon)$ is also included. In Table 1, the present model is denoted by "Hamiltonian" since all flow terms are confined in the Hamiltonian, while "Symplectic" denotes the conventional models whose symplectic part has the flow term. It is seen that both the Hamiltonian and Symplectic models agree with the no flow model in the no flow limit.

3. Push-Forward Representation of General Particle Fluid Moments

3.1 Perturbative expansion of the exact representation

We consider a general scalar particle fluid moment defined by

$$m_{kl}(\boldsymbol{r})\equiv\int\left(\frac{mw^2}{2B}\right)^k v_\parallel^l f\delta^3(\boldsymbol{x}-\boldsymbol{r})\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v}, \tag{16}$$

where f is the particle distribution function and $k,\ l\in\mathbb{N}$. The particle fluid moment can be written in terms of the push-forward transformation associated with the guiding-centre transformation $\mathsf{T}_{\rm GC}^{-1*}$ and the guiding-centre distribution function $F\equiv\mathsf{T}_{\rm GC}^{-1*}f$ as

$$m_{kl}(\boldsymbol{r})=\int\mathrm{d}^6\boldsymbol{Z}\mathcal{J}(\boldsymbol{Z})\left[\mathsf{T}_{\rm GC}^{-1*}\left\{\left(\frac{mw^2}{2B}\right)^k(v_\parallel)^l\right\}\right](\boldsymbol{Z})\times F(\boldsymbol{Z})\delta^3(\mathsf{T}_{\rm GC}^{-1}\boldsymbol{x}-\boldsymbol{r}), \tag{17}$$

where $\mathsf{T}_{\rm GC}^{-1}\boldsymbol{x}=\boldsymbol{X}+\boldsymbol{\rho}+\boldsymbol{\rho}_E+\cdots$ denotes the particle position in the guiding-centre phase space. Push-forward of a scalar function is shown in Fig. 1 schematically. Here we consider a scalar function on z denoted by f and the guiding-centre transformation $\mathsf{T}_{\rm GC}$. We can represent $f(z)$ in terms of $\boldsymbol{Z}$ through $z=\mathsf{T}_{\rm GC}^{-1}\boldsymbol{Z}$ and obtain a function on $\boldsymbol{Z}$, $\mathsf{T}_{\rm GC}^{-1*}f\equiv F$. Since $\mathsf{T}_{\rm GC}^{-1*}$ "pushes forward" f on z to F on $\boldsymbol{Z}$, it is called the push-forward transformation associated with $\mathsf{T}_{\rm GC}$. Note that the action of the transformation is opposite to appearance of the symbol. Conversely, we can obtain a function on z, $\mathsf{T}_{\rm GC}^*G$, from a function on $\boldsymbol{Z}$ denoted by G. Then $\mathsf{T}_{\rm GC}^*$ is called the pull-back transformation associated with $\mathsf{T}_{\rm GC}$.

The formal exact push-forward representation (17) can be explicitly expressed by an infinite series. The infinite series may be approximated by terms at a few orders in ϵ under the condition that the Larmor radius is small compared to the gradient scale length of the fluid moments and the electromagnetic fields. Equation (17) is written approximately as

$$m_{kl}(\boldsymbol{r})\simeq\int\mathrm{d}^6\boldsymbol{Z}\mathcal{J}(\boldsymbol{Z})(\mu-G_1^\mu)^kU^lF(\boldsymbol{Z})\times\delta^3(\boldsymbol{X}+\boldsymbol{\rho}+\boldsymbol{\rho}_E-\boldsymbol{r}), \tag{18}$$

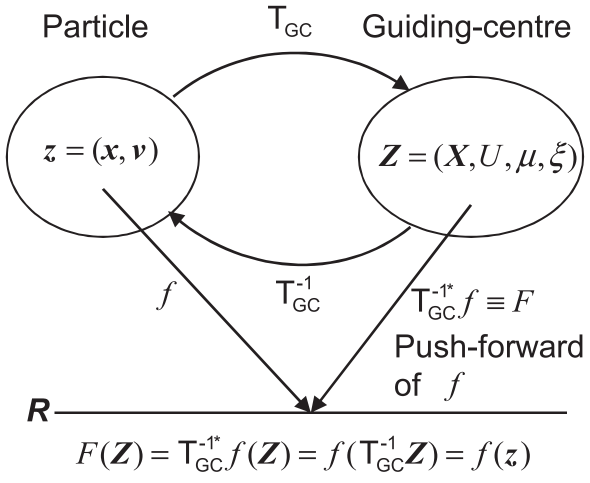

Fig. 1 Push-forward of a scalar function associated with the phase space transformation.

where G_1^μ is kept since it has the nonvanishing gyroaveraged part due to the $\boldsymbol{E}\times\boldsymbol{B}$ flow as mentioned in the previous section. The flow contribution to $\langle G_1^\mu\rangle$ is also found in the Symplectic models for flowing plasmas [5, 6]. This is because the perpendicular particle velocity is measured in a frame moving with flow. Since the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity $\boldsymbol{D}$ is subsonic in most cases, we assume $D\sim\epsilon^{1/2}v_{ti}$ in the following. Expanding the right hand side of the above equation perturbatively, we have the push-forward representation of m_{kl} up to $O(\epsilon^2)$ [31]

$$\begin{aligned}m_{kl}(\boldsymbol{r}) &= M_{kl}(\boldsymbol{r})+\nabla_\perp^2\frac{M_{k+1l}(\boldsymbol{r})}{2q\Omega}\\&+(k+1)\nabla\cdot\left[\frac{M_{kl}(\boldsymbol{r})}{B\Omega}\nabla_\perp\varphi(\boldsymbol{r})\right]\\&-k\boldsymbol{D}(\boldsymbol{r})\cdot\frac{\hat{b}\times\nabla M_{kl}(\boldsymbol{r})}{\Omega},\end{aligned}\tag{19}$$

where the guiding-centre fluid moment M_{kl} is defined by

$$M_{kl}\equiv\int\mu^kU^l\mathcal{J}F\mathrm{d}U\mathrm{d}\mu\mathrm{d}\xi.\tag{20}$$

The last term on the right hand side of Eq. (19) does not appear in the one obtained from the standard gyrokinetic theory in which the lowest order magnetic moment is defined by $mv_\perp^2/2B$ [16]. This term cancels a part of the third term and Eq. (19) is rewritten as

$$\begin{aligned}m_{kl} &\simeq M_{kl}+\frac{1}{2q\Omega}\nabla_\perp^2M_{k+1l}+(k+1)\frac{M_{kl}}{B\Omega}\nabla_\perp^2\varphi\\&+\boldsymbol{D}\cdot\frac{\hat{b}\times\nabla M_{kl}(\boldsymbol{r})}{\Omega},\end{aligned}\tag{21}$$

where terms with ∇B have been neglected since they are of higher order. The last term is rewritten as

$$\boldsymbol{D}\cdot\frac{\hat{b}\times\nabla M_{kl}}{\Omega}=-\boldsymbol{\rho}_E\cdot\nabla M_{kl},\tag{22}$$

which shows the modification to M_{kl} by $\boldsymbol{\rho}_E$. This term, which is also rewritten as $\nabla_\perp\varphi\cdot\nabla M_{kl}/(B\Omega)$, would be comparable to the third term with $\nabla_\perp^2\varphi$ if the gradient scale length of M_{kl} is similar to that of φ. For $k=l=0$, we have the push-forward representation of particle density,

$$\begin{aligned}n &= N+\nabla_\perp^2\frac{P_\perp}{2q\Omega B}+\nabla\cdot\left[\frac{N}{B\Omega}\nabla_\perp\varphi\right]\\&\simeq N+\frac{1}{2q\Omega B}\nabla_\perp^2P_\perp+\frac{N}{B\Omega}\nabla_\perp^2\varphi+\frac{\nabla_\perp\varphi\cdot\nabla N}{B\Omega}\end{aligned}\tag{23}$$

where $n\equiv m_{00}$, $N\equiv M_{00}$ and $P_\perp\equiv BM_{10}$ are particle density, guiding-center density and guiding-centre pependicular pressure, respectively. Difference between m_{kl} and M_{kl} may be important at the tokamak H-mode edge pedestal with the strong radial electric field and the steep radial pressure gradient. For example, the maximum correction to N by the term with $\nabla_\perp^2\varphi$ can reach up to 10% if shear of the radial electric field E_r is $|E_r'|\sim10^7$ V/m^2 and $\Omega\sim10^8$ s^{-1} with $B\sim1$ T which are relevant to the DIII-D H-mode edge [32].

We consider the push-forward repsentation in the Symplectic models for comparison. Since the particle position in the guiding-centre phase space is given by $\mathsf{T}^{-1}_{GC}\boldsymbol{x}\simeq\boldsymbol{X}+\boldsymbol{\rho}$ in the conventional models, the push-forward representation of m_{kl} is written as

$$\begin{aligned}m_{kl}(\boldsymbol{r}) \simeq \int &\mathrm{d}^6\boldsymbol{Z}\mathcal{J}(\boldsymbol{Z})(\mu-G_1^\mu)^kU^l\\&\times F(\boldsymbol{Z})\delta^3(\boldsymbol{X}+\boldsymbol{\rho}-\boldsymbol{r}),\end{aligned}\tag{24}$$

where G_1^μ remains due to the same reason in our model. Under the subsonic ordering, expanding the right hand side of the above equation yields

$$\begin{aligned}m_{kl}(\boldsymbol{r}) &= M_{kl}(\boldsymbol{r})+\nabla_\perp^2\frac{M_{k+1l}(\boldsymbol{r})}{2q\Omega}\\&+k\nabla\cdot\left[\frac{M_{kl}(\boldsymbol{r})}{B\Omega}\nabla_\perp\varphi(\boldsymbol{r})\right]\\&-k\boldsymbol{D}(\boldsymbol{r})\cdot\frac{\hat{b}\times\nabla M_{kl}(\boldsymbol{r})}{\Omega}.\end{aligned}\tag{25}$$

The terms with φ vanish for $k=0$ since $\boldsymbol{\rho}_E$ is not included in the guiding-centre transformation for the guiding-centre position. Hence, the push-forward representation of particle density becomes

$$n=N+\nabla_\perp^2\frac{P_\perp}{2q\Omega B}.\tag{26}$$

Instead, the polarisation drift term appears in the Hamilton equation as seen in Eq. (14).

3.2 Correspondence to the standard gyrokinetic model

The modern standard gyrokinetic model is formulated through the two-step phase space transformation which consists of the guiding-centre transformation T_{GC} and the transformation from the guiding-centre phase space to the

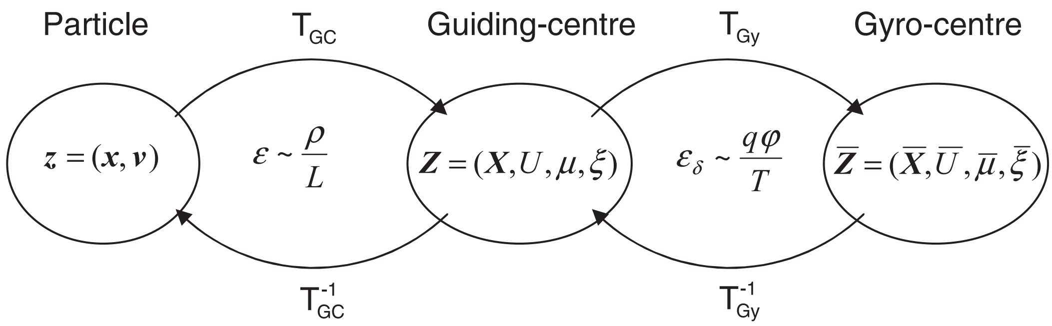

Fig. 2 Phase space transformations in the standard gyrokinetic formulation.

gyro-centre phase space T_{Gy} as shown in Fig. 2 [1]. In the standard formulation, small perturbations of electromagnetic potentials are introduced after the guiding-centre transformation and then the gyro-centre transformation is performed with the small parameter $\epsilon_\delta \sim q\varphi/T \ll 1$. Here we consider a perturbation of the electrostatic potential only. Two exact push-forward representations are possible in the standard gyrokinetic model because of the two phase space transformations. The exact representation usually used is given by [33]

$$m_{kl}(\boldsymbol{r}) = \int \mathrm{d}^6\bar{\boldsymbol{Z}}\mathcal{J}(\bar{\boldsymbol{Z}})\left[\mathsf{T}_{\mathrm{GC}}^{-1*}\left\{\left(\frac{mv_\perp^2}{2B}\right)^k v_\parallel^l\right\}\right](\bar{\boldsymbol{Z}}) \times [\mathsf{T}_{\mathrm{Gy}}^{*}\bar{F}](\bar{\boldsymbol{Z}})\delta^3([\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}](\bar{\boldsymbol{Z}}) - \boldsymbol{r}), \tag{27}$$

where $\bar{\boldsymbol{Z}}$ denotes the gyro-centre coordinates, $\mathcal{J}$ is the Jacobian of the guiding-centre transformation, $\mathsf{T}_{\mathrm{Gy}}^{*}$ is the pull-back transformation associated with T_{Gy}, and $[\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}](\bar{\boldsymbol{Z}}) \simeq \bar{\boldsymbol{X}}+\boldsymbol{\rho}(\bar{\boldsymbol{Z}})$. Recall that originally $\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}$ denotes the particle position in the guiding-centre phase space. $[\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}](\bar{\boldsymbol{Z}})$, however, no longer denotes the particle position. The particle position in the gyro-centre phase space is expressed as $\mathsf{T}_{\mathrm{Gy}}^{-1}\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}$. It is seen that Eq. (27) is the hybrid representation with the push-forward and pull-back transformations. An advantage of the hybrid representation is that effects of the electrostatic potential are contained only in the pull-back of $\bar{F}$, $\mathsf{T}_{\mathrm{Gy}}^{*}\bar{F}$, as

$$\mathsf{T}_{\mathrm{Gy}}^{*}\bar{F} \simeq \bar{F} + \epsilon_\delta\{S_1, \bar{F}\} \simeq \bar{F} + \epsilon_\delta\frac{e\tilde{\varphi}}{B}\frac{\partial\bar{F}}{\partial\bar{\mu}}, \tag{28}$$

where $S_1 = (e/\Omega)\int\tilde{\varphi}\mathrm{d}\bar{\xi}$ is the scalar function generating the gyro-centre transformation at first order, $\tilde{\varphi} = \varphi(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}}) - \langle\varphi(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}})\rangle$ is the gyrophase dependent part of the electrostatic potential, $\bar{\boldsymbol{\rho}} = \boldsymbol{\rho}(\bar{\boldsymbol{Z}})$, and $\langle\cdot\rangle$ denotes the gyrophase average. Although the exact representation (27) gives the push-forward representation similar to Eq. (19) in the appropriate limit as shown in Appendix B, the appearance of Eq. (27) seems to be different from Eq. (17). The incongruity of appearance is resolved by considering the alternative exact representation in the standard gyrokinetics [17],

$$m_{kl}(\boldsymbol{r}) = \int \mathrm{d}^6\bar{\boldsymbol{Z}}\mathcal{J}(\bar{\boldsymbol{Z}})\left[\mathsf{T}_{\mathrm{Gy}}^{-1*}\mathsf{T}_{\mathrm{GC}}^{-1*}\left\{\left(\frac{mv_\perp^2}{2B}\right)^k v_\parallel^l\right\}\right](\bar{\boldsymbol{Z}}) \times \bar{F}(\bar{\boldsymbol{Z}})\delta^3(\mathsf{T}_{\mathrm{Gy}}^{-1}\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x} - \boldsymbol{r}). \tag{29}$$

Similarity between Eqs. (17) and (29) is apparent. The particle position in the gyro-centre phase space, $\mathsf{T}_{\mathrm{Gy}}^{-1}\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}$, is written as

$$\mathsf{T}_{\mathrm{Gy}}^{-1}\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x} = \bar{\boldsymbol{X}} + \epsilon\bar{\boldsymbol{\rho}} + \epsilon_\delta\epsilon\bar{\boldsymbol{\rho}}_{\mathrm{gy}} + \cdots, \tag{30}$$

where $\bar{\boldsymbol{\rho}}_{\mathrm{gy}} = -\{S_1, \bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}}\}$ is the gyro-centre displacement vector associated with the gyro-centre transformation. The gyroaverage of $\bar{\boldsymbol{\rho}}_{\mathrm{gy}}$ corresponds to $\boldsymbol{\rho}_E$ in our model as mentioned before and yields the term with φ in the push-forward representation. It is noted that although $\bar{\boldsymbol{\rho}}_{\mathrm{gy}} \sim O(\epsilon^2)$ under the standard ordering $\epsilon \sim \epsilon_\delta$, $\bar{\boldsymbol{\rho}}_{\mathrm{gy}}$ yields the $O(\epsilon)$ term in the push-forward representation by the assumption that the wavelength of φ is comparable to ρ:

$$\epsilon\epsilon_\delta\int \mathrm{d}^6\bar{\boldsymbol{Z}}\mathcal{J}\bar{F}\bar{\boldsymbol{\rho}}_{\mathrm{gy}}\cdot\bar{\nabla}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r}) = -\epsilon_\delta\int \mathrm{d}^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})\bar{\nabla}\cdot(\bar{\boldsymbol{\rho}}_{\mathrm{gy}}\mathcal{J}\bar{F}).$$

The standard gyrokinetic model is formulated for perturbations with small amplitude and short wavelength [1, 33]. The small amplitude assmuption for the electrostatic potential φ is described as $q\varphi/T \sim \epsilon_\delta \ll 1$ formally. However, φ has the gauge freedom and its amplitude solely has no meaning. Therefore more appropriate assumption is $q\boldsymbol{\rho}\cdot\nabla_\perp\varphi/T \ll 1$ which means that the $\boldsymbol{E}\times\boldsymbol{B}$ drift velocity is much smaller than the thermal velocity. This condition is satisfied not only in the standard gyrokinetic regime with $q\varphi/T \sim \epsilon_\delta$ and $k_\perp\rho \sim 1$ but also in the long wavelength regime with $q\varphi/T \sim 1$ and $k_\perp\rho \sim \epsilon$ where $k_\perp$ is the perpendicular wavenumber of φ. Based on the observation above, it was claimed that the regime of validity of the standard gyrokinetic model can be extended into the long wavelength regime [34]. This interpretation seems to be successfully applied to the gyro-centre transformation for the Hamiltonian or the phase space Lagrangian. It is, however, not necessarily true for the gyrokinetic quasi-neutrality (polarisation) equation and the gyrokinetic Poisson equation either which is usually used to obtain φ. This is because the term with φ in the push-forward representation which results from $\bar{\boldsymbol{\rho}}_{\rm gy}$ goes to higher order for long wavelength. Recall that the push-forward representation of particle density appears in the reduced quasi-neutrality equation and the reduced Poisson equation. If the term from $\bar{\boldsymbol{\rho}}_{\rm gy}$ which is associated with *gyro-centre* transformation goes to $O(\epsilon^2)$, we should also consider the $O(\epsilon^2)$ displacement vector associated with *guiding-centre* transformation in Eq. (30). Generally the guiding-centre transformation for $\boldsymbol{x}$ is written as [1, 19, 21]

$$\boldsymbol{x} = \boldsymbol{X} - \epsilon G_1^X - \epsilon^2 \left(G_2^X - \frac{1}{2} \boldsymbol{G}_1 \cdot \boldsymbol{d} G_1^X \right) + \cdots , \quad (31)$$

where $\boldsymbol{G}_n$ is the nth-order vector field generating the guiding-centre transformation, and $\boldsymbol{G}_n \cdot \boldsymbol{d} = G_n^j \partial_j$. The usual gyroradius vector is given by $\boldsymbol{\rho} = -G_1^X$ in the standard model. Explicit expressions for the other components of $\boldsymbol{G}_1$ and G_2^X are complicated even in the standard model [1, 6]. They are given in Appendix A. Since in the standard gyrokinetic formulation the electric field is not considered in the guiding-centre transformation, the $O(\epsilon^2)$ piece,

$$\boldsymbol{\rho}_B \equiv - \left(G_2^X - \frac{1}{2} \boldsymbol{G}_1 \cdot \boldsymbol{d} G_1^X \right), \quad (32)$$

is related to the nonuniformity of magnetic field only [5]. Hence, we have to keep $\boldsymbol{\rho}_B$ in Eq. (30) when the gradient scale length of the magnetic field L_B is similar to that of the electric field L_E. Besides, spatial variation of $\boldsymbol{\rho}$ should be considered if the gradient scale length of the distribution function L_F is similar to L_B. On the other hand, these pieces can be dropped in large aspect ratio tokamaks and the H-mode edge regions where L_E and L_F are much shorter than L_B.

4. Pull-Back Representation of Guiding-Centre Fluid Moments

Inverse of the push-forward representation is the pull-back representation in which the guiding-centre fluid moments are represented in terms of the particle fluid moments. The pull-back representation is obtained easily if the push-forward representation is known. From Eq. (19) we have

$$\begin{aligned} M_{kl}(\boldsymbol{r}) = m_{kl}(\boldsymbol{r}) - \nabla_\perp^2 \frac{m_{k+1l}(\boldsymbol{r})}{2q\Omega} \\ - (k+1)\nabla \cdot \left[\frac{m_{kl}(\boldsymbol{r})}{B\Omega} \nabla_\perp \varphi(\boldsymbol{r}) \right] \\ + k\boldsymbol{D}(\boldsymbol{r}) \cdot \frac{\hat{b} \times \nabla m_{kl}(\boldsymbol{r})}{\Omega}. \end{aligned} \quad (33)$$

The representation can be also derived from the exact pull-back representation,

$$\begin{aligned} M_{kl}(\boldsymbol{r}) = \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v} \left[\mathsf{T}_{\rm GC}^*(\mu^k U^l) \right] (\boldsymbol{x}, \boldsymbol{v}) \\ \times f(\boldsymbol{x}, \boldsymbol{v}) \delta^3(\mathsf{T}_{\rm GC}\boldsymbol{X} - \boldsymbol{r}), \end{aligned} \quad (34)$$

where $\mathsf{T}_{\rm GC}^*$ is the pull-back transformation associated with the guiding-centre transformation and $\mathsf{T}_{\rm GC}\boldsymbol{X}$ denotes the guiding-centre position in the particle phase space. From the guiding-centre transformation shown in Eqs. (1)-(3), the pull-back representation is written explicitly as

$$\begin{aligned} M_{kl}(\boldsymbol{r}) \simeq \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v} \left(\frac{mw^2}{2B} + G_1^\mu \right)^k v_\parallel^l f \\ \times \delta^3(\boldsymbol{x} - \boldsymbol{\rho} - \boldsymbol{\rho}_E - \boldsymbol{r}). \end{aligned} \quad (35)$$

Under the ordering $D \sim \epsilon^{1/2} v_{ti}$, the pull-back representation of M_{kl} is expanded up to $O(\epsilon^2)$ as

$$\begin{aligned} M_{kl}(\boldsymbol{r}) = m_{kl}(\boldsymbol{r}) + \nabla_\perp^2 \frac{m_{k+1l}(\boldsymbol{r})}{2q\Omega} \\ - (k+1)\nabla \cdot \left[\frac{m_{kl}(\boldsymbol{r})}{B\Omega} \nabla_\perp \varphi(\boldsymbol{r}) \right] \\ + k\boldsymbol{D}(\boldsymbol{r}) \cdot \frac{\hat{b} \times \nabla m_{kl}(\boldsymbol{r})}{\Omega} \\ - \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v} \left(\frac{mw^2}{2B} \right)^k v_\parallel^l f \boldsymbol{\rho} \cdot \nabla \delta^3(\boldsymbol{x} - \boldsymbol{r}). \end{aligned} \quad (36)$$

The last term does not vanish since the particle distribution function is dependent on the gyrophase. Using $\tilde{f} \simeq -\boldsymbol{\rho} \cdot \nabla \langle f \rangle$ [35] and integrating by parts, we have

$$\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v} \left(\frac{mw^2}{2B} \right)^k v_\parallel^l f \boldsymbol{\rho} \cdot \nabla \delta^3(\boldsymbol{x} - \boldsymbol{r}) \simeq \nabla_\perp^2 \frac{m_{k+1l}(\boldsymbol{r})}{q\Omega}, \quad (37)$$

where we have used

$$\langle \boldsymbol{\rho}\boldsymbol{\rho} \rangle = \frac{mw^2}{2B} \frac{\mathsf{I} - \hat{b}\hat{b}}{q\Omega}. \quad (38)$$

Then, Eq. (36) agrees with Eq. (33).

5. Variational Derivation of Push-Forward Representation of Particle Density

5.1 Reduced Vlasov-Poisson variational principle

Although the perturbative expansion of the exact representation is straightforward, as shown in the previous sections, we have to consider the vector field generating the phase space transformation in the expansion. The push-forward representation of particle density can be obtained from the reduced single particle Lagrangian by a variational method in which we need not treat the phase space transformation directly. To this end, we consider a functional derivative of the action functional $I = \int_{t_1}^{t_2} L\mathrm{d}t$ with a Lagrangian for the Vlasov-Poisson system [36],

$$L = \sum_s \int d^6 \boldsymbol{Z}_0 \mathcal{J}_s(\boldsymbol{Z}_0) F_s(\boldsymbol{Z}_0, t_0) \times L_{ps}[\boldsymbol{Z}_s(\boldsymbol{Z}_0, t_0; t), \dot{\boldsymbol{Z}}_s(\boldsymbol{Z}_0, t_0; t), t] - \int d^3\boldsymbol{x} \frac{1}{4\mu_0} \mathsf{F} : \mathsf{F}, \quad (39)$$

where $\sum$ denotes a sum over species, $\boldsymbol{Z}_s(\boldsymbol{Z}_0, t_0; t)$ denotes the guiding-centre coordinates of the particle at t with the initial condition $\boldsymbol{Z}_s(\boldsymbol{Z}_0, t_0; t_0) = \boldsymbol{Z}_0$, μ_0 is permeability of vacuum, the electromagnetic field tensor F is defined by $\mathsf{F}_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$, and $\mathsf{F} : \mathsf{F} \equiv \mathsf{F}^{\mu\nu}\mathsf{F}_{\mu\nu}$. When we use $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$ as the Minkowski spacetime metric, the covariant four vector potential and the four gradient operator are $A_\mu = (-\varphi/c, \boldsymbol{A})$ and $\partial_\mu = ((1/c)\partial_t, \nabla)$, respectively [37][1]. L_{ps} is the reduced single particle Lagrangian of species s such as shown in Eq. (9). $\delta I/\delta\varphi(\boldsymbol{r}) = 0$ yields a reduced Poisson equation,

$$\epsilon_0 \nabla^2 \varphi(\boldsymbol{r}) = \sum_s \int d^6 \boldsymbol{Z} \mathcal{J}_s F_s \frac{\delta L_{ps}}{\delta\varphi(\boldsymbol{r})}, \quad (40)$$

where ϵ_0 is permittivity of vacuum. Recalling the exact Poisson equation,

$$\epsilon_0 \nabla^2 \varphi(\boldsymbol{r}) = -\sum_s q_s n_s, \quad (41)$$

we can regard the right hand side of Eq. (40) as the negative of charge density in terms of the guiding-centre quantities. The requirement that the charge density should agree with $\sum q_s n_s$ yields the push-forward representation of particle density,

$$n(\boldsymbol{r}) = -\frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J}(\boldsymbol{Z}) F(\boldsymbol{Z}) \frac{\delta L_p(\boldsymbol{Z})}{\delta\varphi(\boldsymbol{r})}, \quad (42)$$

where the subscript s is suppressed. This representation is general and valid even if the symplectic part of L_p contains φ as in the Symplectic models [5]. If, as in the present case, φ appears in the Hamiltonian only, the above representation is reduced to

$$n(\boldsymbol{r}) = \frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J}(\boldsymbol{Z}) F(\boldsymbol{Z}) \frac{\delta H(\boldsymbol{Z})}{\delta\varphi(\boldsymbol{r})}. \quad (43)$$

We can obtain an explicit push-forward representation from this equation if the guiding-centre Hamiltonian is known. The push-forward representation of particle density is obtained also by another variational principle with constrained variation [21].

5.2 Push-forward representation of particle density

We can obtain Eq. (23) by considering the following guiding-centre Hamiltonian with subsonic flow,

$$H(\boldsymbol{X}, U, \mu) = \underline{q\varphi(\boldsymbol{X})} + \frac{m}{2}U^2 + \mu B(\boldsymbol{X}) \underline{- \frac{m}{2} D(\boldsymbol{X})^2} + \underline{\frac{m}{2q}\mu \hat{b} \cdot \nabla \times \boldsymbol{D}(\boldsymbol{X})}, \quad (44)$$

which is valid when the gradient scale length of the electric field is much shorter than that of the magnetic field [4]. It is similar to the standard gyrokinetic Hamiltonian in the long wavelength limit. Underlined terms include φ. From the lowest order Hamiltonian $H_0 = q\varphi$, we have the first term on the right hand side of Eq. (23) as

$$\frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J} F \frac{\delta H_0}{\delta\varphi(\boldsymbol{r})} = \int d^6 \boldsymbol{Z} \mathcal{J} F \delta^3(\boldsymbol{X} - \boldsymbol{r}) = N(\boldsymbol{r}). \quad (45)$$

The first order Hamiltonian, $H_1 = mU^2/2 + \mu B - mD^2/2$, yields

$$\frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J} F \frac{\delta H_1}{\delta\varphi(\boldsymbol{r})} = \nabla \cdot \left[\frac{N(\boldsymbol{r})}{B\Omega} \nabla_\perp \varphi(\boldsymbol{r}) \right]. \quad (46)$$

Finally the second order Hamiltonian, $H_2 = (m\mu/2q)\hat{b} \cdot \nabla \times \boldsymbol{D}$, yields

$$\frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J} F \frac{\delta H_2}{\delta\varphi(\boldsymbol{r})} = -\frac{1}{2q} \nabla \cdot \left[\left(\nabla \times \frac{P_\perp}{B} \hat{b} \right) \times \frac{\hat{b}}{\Omega} \right] \simeq \nabla_\perp^2 \frac{P_\perp}{2q\Omega B}, \quad (47)$$

and all terms in Eq. (23) have been derived. It is noted that although the $P_\perp$ term has been approximated in the push-forward representation for comparison with Eq. (23) here, any approximation should be made to the Hamiltonian and the push-forward representation should be derived rigorously from the approximated Hamiltonian for consistency. In the above case, H_2 should be approximated as $H_2 \simeq (\mu/2\Omega)\nabla_\perp^2 \varphi$.

In order to obtain the push-forward representation of particle density in the Symplectic model by the variational method, we must consider the symplectic part of the phase space Lagrangian as well as the Hamiltonian. The guiding-centre Lagrangian in the Symplectic model is given by [6]

$$L_p = \left[q\boldsymbol{A} + m(U\hat{b} + \boldsymbol{D}) \right] \cdot \dot{\boldsymbol{X}} + \frac{m}{q}\mu\dot{\xi} - H, \quad (48)$$

with the Hamiltonian,

$$H = q\varphi + \frac{m}{2}U^2 + \mu B + \frac{m}{2}D^2 + \frac{m}{2q}\mu\hat{b} \cdot \nabla \times \boldsymbol{D}. \quad (49)$$

The only difference in the Hamiltonian from Eq. (44) is the sign in front of $mD^2/2$. The Hamiltonian gives

$$\frac{1}{q} \int d^6 \boldsymbol{Z} \mathcal{J} F \frac{\delta H}{\delta\varphi(\boldsymbol{r})} = N - \nabla \cdot \left[\frac{N}{B\Omega} \nabla_\perp \varphi \right] + \nabla_\perp^2 \frac{P_\perp}{2q\Omega B}, \quad (50)$$

which is the same as the above except the sign in front of the term with φ. In contrast to the Hamiltonian model, the Lagrangian (48) has $\boldsymbol{D}$ in its symplectic part, which yields an additional term in the push-forward representation as

[1] The alternative Minkowski spacetime metric tensor is $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. In this case, the covariant four vector potential is $A_\mu = (\phi/c, -\boldsymbol{A})$ and the four gradient operator is $\partial_\mu = ((1/c)\partial_t, \nabla)$. As a result, the sign of $\mathsf{F}_{\mu\nu}$ flips. Since the sign of $\mathsf{F}^{\mu\nu}$ also flips, $\mathsf{F} : \mathsf{F}$ does not change.

$$n = \int \mathrm{d}^6\boldsymbol{Z}\delta^3(\boldsymbol{X}-\boldsymbol{r})\nabla\cdot\left[\mathcal{J}F\frac{\dot{\boldsymbol{X}}\times\hat{b}}{\Omega}\right] + \frac{1}{q}\int \mathrm{d}^6\boldsymbol{Z}\mathcal{J}F\frac{\delta H}{\delta\varphi(\boldsymbol{r})}. \quad (51)$$

It is found that the first term on the right hand side cancels with the term with φ in the Hamiltonian part by noting $\dot{\boldsymbol{X}}\times\hat{b}\simeq\boldsymbol{D}\times\hat{b}$ and then Eq. (26) is obtained.

Thus the variational method only needs the guiding-centre Hamiltonian (or Lagrangian) and doesn't need details of the guiding-centre transformation. Therefore, this method is very useful in the modern reduced kinetic formulation based on the Lie-transform perturbation analysis. We derive the push-forward representation in the transonic case by the variational method. When the flow speed is comparable to the thermal speed, we have to consider the Hamiltonian,

$$H = q\varphi + \left(\frac{m}{2}U^2 + \mu B - \frac{m}{2}D^2\right) + \frac{1}{2\Omega}\left(\mu + \frac{mD^2}{2B}\right)\nabla_\perp^2\varphi, \quad (52)$$

which is still valid when the gradient scale length of the electric field is much shorter than that of the magnetic field [4]. The Hamiltonian yields the push-forward representaion with additional terms,

$$n = N + \nabla\cdot\nabla_\perp\left[\frac{1}{2q\Omega B}\left(P_\perp + \frac{NmD^2}{2}\right)\right] + \nabla\cdot\left[\left(1-\frac{\nabla_\perp^2\varphi}{2\Omega B}\right)\frac{N}{B\Omega}\nabla_\perp\varphi\right]. \quad (53)$$

The additional terms appear as corrections to the polarisation density. The first one is the flow correction to $P_\perp$. The second one is the correction by the vorticity which gives a term proportional to enstrophy density. They are nonlinear to φ because they come from the cubic terms of φ in the Hamiltonian.

6. Push-Forward Representation of Particle Flux

The push-forward representations of the scalar fluid moments have been discussed in the previous sections. In this section, we consider push-forward representation of a vector fluid moment, a particle flux, by following Refs. [20, 21]. The particle flux is defined in the particle phase space as

$$\boldsymbol{\Gamma}(\boldsymbol{r}) \equiv \int f\boldsymbol{v}\delta^3(\boldsymbol{x}-\boldsymbol{r})\mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{v}. \quad (54)$$

Similar to the scalar fluid moments, $\boldsymbol{\Gamma}(\boldsymbol{r})$ can be expressed as an integral in the guiding-centre phase space,

$$\boldsymbol{\Gamma}(\boldsymbol{r}) = \int \mathrm{d}^6\boldsymbol{Z}\mathcal{J}F\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{v}\delta^3(\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{x}-\boldsymbol{r}), \quad (55)$$

where $\mathsf{T}_{\mathrm{GC}}^{-1}\boldsymbol{v}\simeq\dot{\boldsymbol{X}}+\dot{\boldsymbol{\rho}}_{\mathrm{gc}}$ is push-forward of the particle velocity.

Expanding the delta function in powers of $\boldsymbol{\rho}_{\mathrm{gc}}$ and integrating by parts, we have

$$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_{\mathrm{gc}} + \boldsymbol{\Gamma}_{\mathrm{pol}} + \boldsymbol{\Gamma}_{\mathrm{mag}}, \quad (56)$$

where

$$\boldsymbol{\Gamma}_{\mathrm{gc}} = \int \mathrm{d}^3v\dot{\boldsymbol{X}}F \quad (57)$$

is the guiding-centre flux,

$$\boldsymbol{\Gamma}_{\mathrm{pol}} = \frac{\partial}{\partial t}\int \mathrm{d}^3v\boldsymbol{\rho}_{\mathrm{gc}}F \quad (58)$$

is the polarisation flux,

$$\boldsymbol{\Gamma}_{\mathrm{mag}} = \nabla\times\left\{\int \mathrm{d}^3vF\left[\boldsymbol{\rho}_{\mathrm{gc}}\times\left(\frac{1}{2}\dot{\boldsymbol{\rho}}_{\mathrm{gc}}+\dot{\boldsymbol{X}}\right)\right]\right\} \quad (59)$$

is the magnetisation flux, and $\mathrm{d}^3v = \mathcal{J}\mathrm{d}U\mathrm{d}\mu\mathrm{d}\xi$. In the Symplectic models, $\boldsymbol{\rho}_{\mathrm{gc}}$ is the usual Larmor radius vector $\boldsymbol{\rho}$. Then $\langle\boldsymbol{\rho}_{\mathrm{gc}}\rangle = 0$ and $\boldsymbol{\Gamma}_{\mathrm{pol}}$ vanishes. Instead, as shown in Eq. (14), the polarisation drift term

$$\boldsymbol{V}_{\mathrm{pol}} = \frac{\hat{b}}{\Omega}\times\frac{\partial\boldsymbol{D}}{\partial t} \quad (60)$$

is included in the guiding-centre drift $\dot{\boldsymbol{X}}$ and it yields the polarisation flux. On the other hand, $\dot{\boldsymbol{X}}$ in the Hamiltonian model does not include the polarisation drift $\boldsymbol{V}_{\mathrm{pol}}$. This is because of the difference in the symplectic part of the guiding-centre Lagrangian mentioned before. Recall that the purpose of the Hamiltonian model is to exclude the time derivative terms from the guiding-centre Hamilton equations. In the Hamiltonian model, $\boldsymbol{\rho}_{\mathrm{gc}}$ is not purely oscillatory and $\langle\boldsymbol{\rho}_{\mathrm{gc}}\rangle = \boldsymbol{\rho}_E$. Then $\boldsymbol{\Gamma}_{\mathrm{pol}}$ becomes

$$\boldsymbol{\Gamma}_{\mathrm{pol}} = \frac{\hat{b}}{\Omega}\times\frac{\partial}{\partial t}(N\boldsymbol{D}). \quad (61)$$

Besides, while the second part of $\boldsymbol{\Gamma}_{\mathrm{mag}}$ including $\boldsymbol{\rho}_{\mathrm{gc}}\times\dot{\boldsymbol{X}}$ also vanishes due to $\langle\boldsymbol{\rho}_{\mathrm{gc}}\rangle = 0$ in the Symplectic models, it does not in the Hamiltonian model.

7. Summary

We have considered the push-forward representation of fluid moments for the two basic types (Hamiltonian and Symplectic [1]) of guiding-centre models for plasmas with large flow velocity. In each case, variation of the system Lagrangian with respect to the electrostatic potential yields the push-forward representation of particle density in a unique way.

The explicit representations which depend on details of the guiding-centre transformations have been derived from the exact representation by the perturbative expansion in the subsonic flow case. The representation in the Hamiltonian model is similar to that in the standard gyrokinetic model in the long wavelength limit since the symplectic part of the phase space Lagrangian is common. Use of the relative perpendicular particle velocity to the flow for the definition of the magnetic moment causes the additional flow term in the representation. However, this term does not appear in the representation of particle density. The representation in the Symplectic model also has the additional flow term due to the same reason, while the representation of particle density in the Symplectic model has no polarisation density term with the electrostatic potential which appears in the Hamiltonian model and in the standard gyrokinetic model. This is due to the difference in the transformation for the guiding-centre position. Although the appearance of the exact representation usually used in the standard gyrokinetic formulation is different from that in the Hamiltonian model for flowing plasmas, the correspondence between the two models becomes apparent by considering the alternative exact representation in the standard gyrokinetic model. Besides, the observation of the alternative representation shows that the second order displacement vector associated with the guiding-centre transformation should be considered on an equal footing as the gyro-centre displacement vector in the push-forward representation in the gyrokinetic model when the gradient scale length of the electrostatic potential is comparable to that of the magnetic field. The push-forward representation of particle density can also be obtained from the guiding-centre Lagrangian (or Hamiltonian) by the variational method. The variational method confirms the uniqueness of the representation for any particular Lagrangian. This method is usuful in the modern reduced kinetic formulation based on the Lie-transform perturbation method. The results obtained by the perturbative expansion are recovered from the appropriate Lagrangians. The representation of particle density in the transonic case has been obtained by the variational method and the corrections to the polarisation density have been found.

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Appendix A. Vector Field Generating the Guiding-Centre Transformation

The components of the first order generating vector field are

$$G_1^X = -\boldsymbol{\rho} - \boldsymbol{\rho}_E, \tag{A.1}$$

$$\begin{aligned} G_1^U = {}& \frac{\mu}{q}(\mathsf{a}_1 : \nabla\hat{b} + \hat{b}\cdot\nabla\times\hat{b}) \\ &+ \frac{1}{2}\frac{W}{\Omega}\hat{c}\cdot\nabla\times(\boldsymbol{D}^* + U\hat{b}) \\ &+ \frac{1}{2\Omega}\boldsymbol{D}\cdot\nabla\times(\boldsymbol{D}^* + U\hat{b}) \\ &+ \frac{W}{2B\Omega}\nabla_\perp\varphi\cdot\nabla\hat{b}\cdot\hat{c} + \frac{1}{2}\hat{b}\cdot\boldsymbol{F}, \end{aligned} \tag{A.2}$$

$$\begin{aligned} G_1^\mu = {}& -\mu G_1^X\cdot\nabla\log B - \frac{m}{2B}\boldsymbol{D}\cdot\boldsymbol{F} \\ &+ \frac{m}{2\Omega}\boldsymbol{D}\cdot W\hat{c}\frac{\hat{b}}{2B}\cdot\nabla\times\left(\boldsymbol{D}^* + U\hat{b}\right) \\ &- \frac{m}{2B^2\Omega}\nabla_\perp\varphi\cdot\nabla\boldsymbol{D}^*\cdot W\hat{c} - \frac{\mu}{\Omega}\boldsymbol{a}_1 : \nabla\boldsymbol{D}^* \\ &- \frac{m}{B\Omega}W\hat{c}\times\hat{b}\cdot\frac{\partial\boldsymbol{D}}{\partial t} \\ &- \frac{mU}{B}\left[\frac{W\hat{c}}{2\Omega}\cdot\nabla\times\left(\boldsymbol{D}^* + U\hat{b}\right) + \frac{1}{2}\hat{b}\cdot\boldsymbol{F}\right] \\ &- \frac{7}{6}\frac{\mu}{\Omega}(\hat{b}\times\boldsymbol{D})\cdot\nabla\log B - \frac{\mu}{\Omega}\hat{b}\cdot\nabla\times\boldsymbol{D}^*, \end{aligned} \tag{A.3}$$

$$G_1^\xi = G_1^X\cdot\boldsymbol{R} + \frac{q}{3}\frac{\partial G_1^X}{\partial\mu}\cdot\boldsymbol{F} - \frac{q}{m}\frac{\partial S_3'}{\partial\mu}, \tag{A.4}$$

$$\begin{aligned} S_3' = {}& \frac{m}{q}\Bigg[-\frac{m}{2B}\boldsymbol{D}^*\cdot\hat{\boldsymbol{F}} \\ &+ \frac{m}{4B\Omega}\{(\boldsymbol{D}\cdot W\hat{a})\hat{b} - 2UW\hat{a}\}\nabla\times\left(\boldsymbol{D}^* + U\hat{b}\right) \\ &- \frac{m}{2B^2\Omega}\nabla_\perp\varphi\cdot\nabla\boldsymbol{D}^*\cdot W\hat{a} \\ &- \frac{\mu}{\Omega}\boldsymbol{a}_2 : \nabla\left(\boldsymbol{D}^* + \frac{2}{3}\boldsymbol{D}\right) - \frac{mW}{B\Omega}\hat{c}\cdot\frac{\partial\boldsymbol{D}}{\partial t} \\ &+ \frac{\mu}{2B\Omega}\boldsymbol{a}_1 : (\nabla\varphi)(\nabla\log B) - \frac{2}{3}\frac{W}{\Omega}\mu\hat{c}\cdot\nabla\log B \\ &+ \frac{m}{3B\Omega}(\boldsymbol{D}\cdot W\hat{a})\hat{b}\cdot\nabla\times\left(\boldsymbol{D}^* + \frac{1}{2}U\hat{b}\right)\Bigg], \end{aligned} \tag{A.5}$$

where $W = (2\mu B/m)^{1/2}$, $\boldsymbol{R} = (\nabla\hat{c})\cdot\hat{a}$, $\hat{a} = \hat{b}\times\hat{c}$, $\boldsymbol{D}^* = \boldsymbol{D} + U\hat{b}$,

$$\mathsf{a}_1 = -\frac{\hat{a}\hat{c} + \hat{c}\hat{a}}{2}, \quad \mathsf{a}_2 = \frac{\hat{c}\hat{c} - \hat{a}\hat{a}}{4}, \tag{A.6}$$

$$\boldsymbol{F} = \frac{3}{2}(\boldsymbol{D}\cdot\boldsymbol{\rho})\nabla\log B - (\nabla\boldsymbol{D})\cdot\boldsymbol{\rho}, \tag{A.7}$$

$$\hat{\boldsymbol{F}} = -\frac{3}{2}\frac{W}{\Omega}\boldsymbol{D}\cdot\hat{c}\nabla\log B + (\nabla\boldsymbol{D})\cdot\frac{W}{\Omega}\hat{c}. \tag{A.8}$$

The spatial component of the second order vector field is given by

$$\begin{aligned} G_2^X = {}& \frac{1}{2}\left(g_1^\mu\frac{\partial\boldsymbol{\rho}}{\partial\mu} + g_1^\xi\frac{\partial\boldsymbol{\rho}}{\partial\xi} - \frac{\hat{b}\times\boldsymbol{F}}{\Omega}\right) \\ &- \frac{G_1^X}{2\Omega}\hat{b}\cdot\nabla\times(\boldsymbol{D}^* + U\hat{b}) - \frac{1}{m}\frac{\partial S_3'}{\partial U}\hat{b}, \end{aligned} \tag{A.9}$$

where $g_1^\mu = G_1^\mu + \mu G_1^X\cdot\nabla\log B$ and $g_1^\xi = G_1^\xi - G_1^X\cdot\boldsymbol{R}$. When there is no electric field, they reduce to the standard results [1, 38]:

$$G_1^X = -\boldsymbol{\rho}, \tag{A.10}$$

$$G_1^U = \frac{\mu}{q}(\mathsf{a}_1 : \nabla\hat{b} + \hat{b}\cdot\nabla\times\hat{b}) + \frac{UW}{\Omega}\hat{c}\cdot\nabla\times\hat{b}, \tag{A.11}$$

$$\begin{aligned} G_1^\mu = {}& \mu\boldsymbol{\rho}\cdot\nabla\log B - \frac{U\mu}{\Omega}\mathsf{a}_1 : \nabla\hat{b} \\ &- \frac{mU^2}{B}\frac{W}{\Omega}\hat{c}\cdot\nabla\times\hat{b} - \frac{U\mu}{\Omega}\hat{b}\cdot\nabla\times\hat{b}, \end{aligned} \tag{A.12}$$

$$G_1^\xi = -\boldsymbol{\rho}\cdot\boldsymbol{R} + \frac{mU^2}{2\mu B}\frac{W}{\Omega}\hat{a}\cdot\nabla\times\hat{b} + \frac{U}{\Omega}\mathsf{a}_2 : \nabla\hat{b} + \frac{W}{\Omega}\hat{c}\cdot\nabla\log B, \tag{A.13}$$

$$S_3' = \frac{m}{q}\left[-\frac{mU^2W}{B\Omega}\hat{a}\cdot\nabla\times\hat{b} - \frac{U\mu}{\Omega}\mathsf{a}_2 : \nabla\hat{b} - \frac{2}{3}\frac{W}{\Omega}\mu\hat{c}\cdot\nabla\log B\right], \tag{A.14}$$

and

$$G_2^X = \frac{1}{2}\left(g_1^\mu\frac{\partial\boldsymbol{\rho}}{\partial\mu} + g_1^\xi\frac{\partial\boldsymbol{\rho}}{\partial\xi}\right) + \boldsymbol{\rho}\left(\frac{U}{\Omega}\hat{b}\cdot\nabla\times\hat{b}\right) - \frac{1}{m}\frac{\partial S_3'}{\partial U}\hat{b}. \tag{A.15}$$

Then, the second order guiding-centre displacement vector $\boldsymbol{\rho}_B$ is given by [5]

$$\begin{aligned}\boldsymbol{\rho}_B = -\frac{1}{\Omega^2}\Bigg[&U^2\hat{b}\cdot\nabla\hat{b} \\ &+ UW\left\{\frac{1}{2}(\hat{b}\cdot\nabla\times\hat{b} - \mathsf{a}_1 : \nabla\hat{b})\hat{a} + 2(\hat{b}\cdot\nabla\hat{b}\cdot\hat{c})\hat{b} + (\mathsf{a}_2 : \nabla\hat{b})\hat{c}\right\} \\ &+ \frac{\mu B}{m}\left\{\left(\frac{1}{2}\nabla\cdot\hat{b} - \mathsf{a}_2 : \nabla\hat{b}\right)\hat{b} + \frac{3}{2}\nabla_\perp\log B + 2\mathsf{a}_2\cdot\nabla\log B\right\}\Bigg].\end{aligned} \tag{A.16}$$

Appendix B. Push-Forward Representation in the Standard Gyrokinetic Model

The formal exact representation in the standard gyrokinetic model (27) is written approximately as

$$m_{kl}(\boldsymbol{r}) = \int d^6\bar{\boldsymbol{Z}}\mathcal{J}(\bar{\boldsymbol{Z}})\bar{\mu}^k\bar{U}^l\left[\bar{F} + \epsilon_\delta\frac{q\tilde{\varphi}}{B}\frac{\partial\bar{F}}{\partial\bar{\mu}}\right]\times\delta^3(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}} - \boldsymbol{r}), \tag{B.1}$$

where $\tilde{\varphi} = \varphi(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}}) - \langle\varphi(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}})\rangle$. Noting

$$\varphi(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}}) = \exp(\bar{\boldsymbol{\rho}}\cdot\bar{\nabla})\varphi(\bar{\boldsymbol{X}}),$$
$$\delta^3(\bar{\boldsymbol{X}} + \bar{\boldsymbol{\rho}} - \boldsymbol{r}) = \exp(\bar{\boldsymbol{\rho}}\cdot\bar{\nabla})\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r}),$$

we have

$$\begin{aligned} m_{kl}(\boldsymbol{r}) &= \int d^6\bar{\boldsymbol{Z}}\mathcal{J}(\bar{\boldsymbol{Z}})\bar{\mu}^k\bar{U}^l \times\left[\bar{F} + \epsilon_\delta\frac{q}{B}(e^{\bar{\rho}\cdot\bar{\nabla}}\varphi(\bar{\boldsymbol{X}}) - \langle e^{\bar{\rho}\cdot\bar{\nabla}}\varphi(\bar{\boldsymbol{X}})\rangle)\frac{\partial\bar{F}}{\partial\bar{\mu}}\right] \times e^{\bar{\rho}\cdot\bar{\nabla}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r}) \\ &= \int d^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})e^{-\bar{\rho}\cdot\bar{\nabla}}\Bigg[\mathcal{J}(\bar{\boldsymbol{Z}})\bar{\mu}^k\bar{U}^l \times\left\{\bar{F} + \epsilon_\delta\frac{q}{B}(e^{\bar{\rho}\cdot\bar{\nabla}}\varphi(\bar{\boldsymbol{X}}) - \langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle\varphi(\bar{\boldsymbol{X}}))\frac{\partial\bar{F}}{\partial\bar{\mu}}\right\}\Bigg] \\ &= \bar{M}_{kl}(\boldsymbol{r}) + \epsilon_\delta\int d^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})e^{-\bar{\rho}\cdot\bar{\nabla}} \times\left[\mathcal{J}(\bar{\boldsymbol{Z}})\bar{\mu}^k\bar{U}^l\frac{q}{B}\frac{\partial\bar{F}}{\partial\bar{\mu}}(e^{\bar{\rho}\cdot\bar{\nabla}}\varphi(\bar{\boldsymbol{X}}) - \langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle\varphi(\bar{\boldsymbol{X}}))\right]\end{aligned} \tag{B.2}$$

where spatial variation of $\bar{\boldsymbol{\rho}}$ has been neglected. The first term is a gyroaveraged gyrofluid moment defined by [7,14]

$$\bar{M}_{kl}(\bar{\boldsymbol{X}}) \equiv \int d\bar{U}d\bar{\mu}d\bar{\xi}e^{-\bar{\rho}\cdot\bar{\nabla}}\mathcal{J}(\bar{\boldsymbol{Z}})\bar{\mu}^k\bar{U}^l\bar{F}, \tag{B.3}$$

which is represented in terms of gyrofluid moments as

$$\bar{M}_{kl} = M_{kl} + \nabla_\perp^2\frac{M_{k+1l}}{2q\Omega} + \cdots, \tag{B.4}$$

where M_{kl} is the gyrofluid moment defined by

$$M_{kl}(\bar{\boldsymbol{X}}) \equiv \int\bar{\mu}^k\bar{U}^l\mathcal{J}(\bar{\boldsymbol{Z}})\bar{F}(\bar{\boldsymbol{Z}})d\bar{U}d\bar{\mu}d\bar{\xi}. \tag{B.5}$$

When $\bar{F}$ is approximated by a Maxwellian $F_M \propto \exp(-\bar{\mu}B/T_\perp)$, we have

$$\frac{\partial\bar{F}}{\partial\bar{\mu}} = -\frac{B}{T_\perp}F_M, \tag{B.6}$$

which is widely used in gyrokinetic and gyrofluid models. Using this approximation and neglecting action of the operator $\exp(-\bar{\boldsymbol{\rho}}\cdot\bar{\nabla})$ on F_M, B and $T_\perp$, the $O(\epsilon_\delta)$ part in Eq. (B.2) is reduced as

$$\begin{aligned}&\int d^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})e^{-\bar{\rho}\cdot\bar{\nabla}} \times\left[\mathcal{J}\bar{\mu}^k\bar{U}^l\frac{q}{B}\frac{\partial\bar{F}}{\partial\bar{\mu}}(e^{\bar{\rho}\cdot\bar{\nabla}}\varphi(\bar{\boldsymbol{X}}) - \langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle\varphi(\bar{\boldsymbol{X}}))\right] \\ &= \int d^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})\mathcal{J}\bar{\mu}^k\bar{U}^l\frac{q}{T_\perp}F_M \times\left(e^{-\bar{\rho}\cdot\bar{\nabla}}\langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle\varphi(\bar{\boldsymbol{X}}) - \varphi(\bar{\boldsymbol{X}})\right) \\ &= \int d^6\bar{\boldsymbol{Z}}\delta^3(\bar{\boldsymbol{X}} - \boldsymbol{r})\mathcal{J}\bar{\mu}^k\bar{U}^l\frac{q}{T_\perp}F_M\left(\langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle^2 - 1\right)\varphi(\bar{\boldsymbol{X}}),\end{aligned} \tag{B.7}$$

where we have used $\langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle = \langle e^{-\bar{\rho}\cdot\bar{\nabla}}\rangle$. For $k = l = 0$ with the Maxwellian approximation in the $O(\epsilon_\delta)$ part, we have

$$n(\boldsymbol{r}) = \bar{N}(\boldsymbol{r}) + n_0(\Gamma_0 - 1)\frac{q\varphi(\boldsymbol{r})}{T_\perp}, \tag{B.8}$$

where $n_0 = \int F_M\mathcal{J}d\bar{U}d\bar{\mu}d\bar{\xi}$ and Γ_0 is the operator defined by

$$\Gamma_0 \equiv \frac{1}{n_0}\int d\bar{U}d\bar{\mu}d\bar{\xi}\mathcal{J}F_M\langle e^{\bar{\rho}\cdot\bar{\nabla}}\rangle^2. \tag{B.9}$$

In the wavenumber space this operator becomes $I_0(b)e^{-b}$ in which $I_0(b)$ is the modified Bessel function of the zeroth kind and its argument is $b = k_\perp^2\rho^2$ with the thermal gyroradius $\rho = v_t/\Omega$. When the gyroradius for electrons is neglected, electron guiding-centre density coincides with the electron particle density. Then, the above push-forward representation of particle density can be interpreted as the gyrokinetic or gyrofluid quasi-neutrality condition between electrons and singly charged ions,

$$n_e = \bar{N}_i + n_{i0}(\Gamma_0 - 1)\frac{e\varphi}{T_{\perp i}}. \tag{B.10}$$

This equation is also derived by variational methods [36, 39, 40]. When we take the long wavelength limit, but not approximate $\partial\bar{F}/\partial\bar{\mu}$ by $(-B/T_\perp)F_M$, we have

$$m_{kl}(\boldsymbol{r}) = M_{kl}(\boldsymbol{r}) + \nabla_\perp^2 \frac{M_{k+1l}(\boldsymbol{r})}{2q\Omega} + (k+1)\nabla\cdot\left[\frac{M_{kl}(\boldsymbol{r})}{B\Omega}\nabla_\perp\varphi(\boldsymbol{r})\right], \tag{B.11}$$

where we have assumed that $(k_\perp\rho)^2 \sim \epsilon_\perp$ for a small $O(\epsilon_\delta)$ perturbation, $(k_\perp\rho) \sim \epsilon_\perp$ for a $O(1)$ moment and $\epsilon \sim \epsilon_\delta \sim \epsilon_\perp$. For $k = l = 0$, we have the push-forward representation of the particle density n,

$$n = N + \nabla_\perp^2 \frac{P_\perp}{2q\Omega B} + \nabla\cdot\left[\frac{N}{\Omega B}\nabla_\perp\varphi\right], \tag{B.12}$$

where N and $P_\perp$ are the gyro-centre density and the gyro-centre perpendicular pressure defined by

$$N \equiv \int \bar{F}\mathcal{J}\mathrm{d}\bar{U}\mathrm{d}\bar{\mu}\mathrm{d}\bar{\xi}, \tag{B.13}$$

$$P_\perp \equiv \int \bar{\mu}B\bar{F}\mathcal{J}\mathrm{d}\bar{U}\mathrm{d}\bar{\mu}\mathrm{d}\bar{\xi}, \tag{B.14}$$

respectively.

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