# Preliminary Study of Uncertainty-Driven Plasma Diffusion

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Quantum mechanical plasma diffusion is studied using a semi-classical model with two different characteristic lengths; one is the average interparticle separation, and the other is the magnetic length. The diffusion coefficients $D$ derived in this study show a dependence on several plasma parameters, such as temperature $T$, mass $m$, density $n$, and magnetic field $B$, similar to that observed experimentally. The numerical values of the diffusion coefficient $D$ in this study are as large as that for neoclassical diffusion. We have pointed out in this study that (i) for distant encounters in typical fusion plasmas of $T = 10\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$, the average potential energy $\langle U \rangle \sim 30\,\mathrm{meV}$ is as small as the uncertainty in energy $\Delta E \sim 40\,\mathrm{meV}$, and (ii) for a magnetic field $B = 3\,\mathrm{T}$, the spatial size of the wavefunction in the plane perpendicular to the magnetic field is as large as $\ell_\mathrm{B} \sim 1.4 \times 10^{-8}\,\mathrm{m}$, which is much larger than the typical electron wavelength $\lambda_\mathrm{e} \sim 10^{-11}\,\mathrm{m}$.



Keywords: magnetic length, uncertainty principle, quantum mechanical diffusion, anomalous diffusion, neoclassical theory.

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## 1. Introduction

More than half a century ago, it was noted that to correctly analyze diffusion of plasmas, one must consider the wave character of charged particles when the temperature $T$ is high, i.e., the relative speeds of interacting particles are high [1, 2]. The criterion for the classical theory to be valid in terms of relative speed $g$ in a hydrogen plasma is given in Ref. [2] as

$$g \ll \frac{2e^2}{4\pi\epsilon_0\hbar} = 4.4 \times 10^6\,\mathrm{m/s}, \tag{1}$$

where $e = 1.60 \times 10^{-19}$ C, and $\hbar \equiv h/2\pi = 1.05 \times 10^{-34}$ J·s represent the elementary electric charge and Planck constant, respectively. In contemporary fusion plasmas with $T \sim 10\,\mathrm{keV}$ or higher, ions as well as electrons should be treated quantum mechanically. In current plasma physics, however, the quantum mechanical effects enter as a minor correction to the Coulomb logarithm, $\ln\Lambda$, in the case of close encounters [3]. Nonetheless, the neoclassical theory is capable of predicting many phenomena, such as those related to current conduction. Such phenomena depend linearly on the change in velocity $\Delta\boldsymbol{v}$ or in position $\Delta\boldsymbol{r}$. The quantum mechanical changes, e.g., in the velocity ${}^{\mathrm{QM}}\Delta\boldsymbol{v}$, are stochastic. The average or expectation value of $\Delta\boldsymbol{v}$ conforms to the classical prediction ${}^{\mathrm{CL}}\Delta\boldsymbol{v}$ due to the Ehrenfest theorem: for $\boldsymbol{\xi} = \boldsymbol{v}, \boldsymbol{r}$

$$\langle \Delta\boldsymbol{\xi} \rangle = \left\langle {}^{\mathrm{CL}}\Delta\boldsymbol{\xi} + {}^{\mathrm{QM}}\Delta\boldsymbol{\xi} \right\rangle = {}^{\mathrm{CL}}\Delta\boldsymbol{\xi}. \tag{2}$$

However, diffusion is quadratic in $\Delta\boldsymbol{g}$ or $\Delta\boldsymbol{r}$:

$$\left\langle (\Delta\boldsymbol{\xi})^2 \right\rangle = \left({}^{\mathrm{CL}}\Delta\boldsymbol{\xi}\right)^2 + \left\langle \left({}^{\mathrm{QM}}\Delta\boldsymbol{\xi}\right)^2 \right\rangle > \left({}^{\mathrm{CL}}\Delta\boldsymbol{\xi}\right)^2. \tag{3}$$

This might explain why we cannot understand so-called anomalous diffusion using classical theories that give only the correct $\langle \Delta\boldsymbol{\xi} \rangle$.

## 2. Uncertainties in Plasmas

A classical particle obeys the deterministic equation of motion, which gives the particle's trajectory in phase space $(\boldsymbol{r}, \boldsymbol{v})$ at a time $t$. The actual *trajectory* of a particle with mass $m$, however, is stochastic in the phase space with uncertainties in position $\Delta\boldsymbol{r}$, in velocity $\Delta\boldsymbol{v}$, and in energy $\Delta E$ in a time interval $\Delta t$ because of the uncertainty relation:

$$\Delta r \Delta v > \frac{\hbar}{m}, \quad \Delta E > \frac{\hbar}{\Delta t}. \tag{4}$$

Equation (4) tells us that (i) a lighter particle has a larger uncertainty in phase space, and (ii) the uncertainty in energy $\Delta E$ is larger for shorter time intervals. Since, for a given time interval $\Delta t$, there are three unknowns in Eq. (4), $\Delta r$, $\Delta v$, and $\Delta E$, we need to find or impose another relationship among these uncertainties. For this purpose, let $L$ be the length the particle travels during some characteristic time interval, i.e., $L \equiv v_0 \Delta t$, where $v_0$ is the initial particle speed.

In the presence of a static magnetic field $\boldsymbol{B}$, the classical particle's energy $E = m\boldsymbol{v}^2/2$ is a constant of the motion:

$$\Delta E = m\Delta\boldsymbol{v} \cdot \left(\boldsymbol{v}_0 + \frac{\Delta\boldsymbol{v}}{2}\right) = 0. \tag{5}$$

The same relation as Eq. (5), $\Delta E = 0$, holds for the relative velocity $\boldsymbol{g} = \boldsymbol{v}_i - \boldsymbol{v}_j$ and its change $\Delta\boldsymbol{g}$ in binary Coulomb interactions in the absence of a magnetic field. For a quantum mechanical particle, $\Delta E$ is not necessarily zero due to

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the uncertainty relation:

$$\Delta E \sim m\boldsymbol{v}_0 \cdot \Delta \boldsymbol{v} > \frac{\hbar}{\Delta t}. \tag{6}$$

Comparing Eq. (6) with the uncertainty relation in Eq. (4), we have

$$\Delta r \sim L, \quad \Delta v > \frac{\hbar}{mL}. \tag{7}$$

Thus, the square of the uncertainty in the cyclotron center $\boldsymbol{r}_G = \boldsymbol{r} + \boldsymbol{v} \times \boldsymbol{\omega}/\omega^2$, where $\boldsymbol{\omega} = q\boldsymbol{B}/m$ is the cyclotron frequency vector, is given by

$$(\Delta \boldsymbol{r}_G)^2 = (\Delta \boldsymbol{r})^2 + \left(\frac{m\Delta \boldsymbol{v}}{qB}\right)^2 \sim L^2 + \left(\frac{\ell_B^2}{L}\right)^2, \tag{8}$$

where $\ell_B = \sqrt{\hbar/qB}$ is called the magnetic length in quantum mechanics.

### 2.1 Uncertainties in the presence of a $B$ field

Note that the magnetic length $\ell_B = \sqrt{\hbar/qB}$ is the spatial size of a wave packet in the plane perpendicular to the magnetic field [4], i.e.,

$$|\psi(\boldsymbol{r}_\perp, t)|^2 = \frac{1}{\pi\ell_B^2} \exp\left[-\frac{(\boldsymbol{r}_\perp - \langle \boldsymbol{r}_\perp(t)\rangle)^2}{\ell_B^2}\right], \tag{9}$$

where $\psi(\boldsymbol{r}_\perp, t)$ stands for the wavefunction, and $\langle \boldsymbol{r}_\perp(t)\rangle$ is the classical position of the particle in the plane perpendicular to $\boldsymbol{B}$. The square of the cyclotron radius $\rho^2$ and the energy $E$ are quantized with the energy levels, i.e., the Landau levels [4], of $N = 0, 1, 2, \ldots,$ as follows:

$$\rho_N^2 = (2N + 1)\,\ell_B^2, \tag{10}$$

$$E_N = (N + 1/2)\,\hbar\omega. \tag{11}$$

Thus, the magnetic length $\ell_B$ is the cyclotron radius in the ground state, $N = 0$. For example, for a magnetic field $B = 3\,\mathrm{T}$, the spatial size of the wavefunction in the plane perpendicular to the magnetic field is as large as $\ell_B \sim 1.4\times10^{-8}$ m. This length is comparable to the average interparticle separation $\Delta\ell \sim 2\times10^{-7}$ m and is much larger than the typical wavelength of electrons, $\lambda_e \sim 10^{-11}$ m, in a fusion plasma with a temperature $T \sim 10\,\mathrm{keV}$ and a number density $n = 10^{20}\,\mathrm{m}^{-3}$. Thus, a charged particle behaves quantum mechanically for a time interval $\Delta t = \ell_B/v_0$, i.e., $L = \ell_B$ in the presence of a magnetic field.

### 2.2 Uncertainties in the absence of a $B$ field

In the absence of a magnetic field, $\boldsymbol{B} = 0$, choosing $L = \Delta\ell \equiv n^{-1/3}$, where $\Delta\ell$ stands for the average interparticle separation, and $n$ is the number density of the plasma, the uncertainty in energy becomes

$$\Delta E > \frac{hg_{th}}{\Delta\ell} \sim 40\,\mathrm{meV} \tag{12}$$

for protons in a hydrogen plasma of $T = 10\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$, where $g_{th} \equiv \sqrt{4T/m_p} \sim 2\times10^6$ m/s is the relative thermal speed, with $m_p$ being the proton mass. On the other hand, the average potential energy $U$ of a particle in the plasma is approximately given by

$$\langle U\rangle \sim U(\Delta\ell/4) \sim 30\,\mathrm{meV}, \tag{13}$$

where $U(r) = e^2/4\pi\epsilon_0 r$, $r$ stands for the interparticle separation, and we have used $\langle 1/r\rangle \sim 4/\Delta\ell$. Thus, the potential energy of a particle is, on the average, as small as the uncertainty in energy in typical fusion plasma. This means that binary interaction, especially diffusion, in plasmas might not be governed by the Coulomb potential but by the quantum mechanical uncertainty: $\langle U\rangle \sim \Delta E$. For this reason we will select the characteristic lengths $L = \Delta\ell$ and $L = \ell_B$ in the following section.

## 3. Semiclassical Model for Motion

Let us assume that a particle with a positive charge $q > 0$ is moving in the presence of a uniform magnetic field $\boldsymbol{B} = (0, 0, B)$ in the $z$-direction. First, we integrate the equation of motion for the classical particle for the time interval $\Delta t$ to get the classical position in the phase space $(\boldsymbol{r}(\Delta t), \boldsymbol{v}(\Delta t))$,

$$\boldsymbol{r}(\Delta t) = \boldsymbol{r}(0) + \int_0^{\Delta t} \boldsymbol{v}(t)\ \mathrm{d}t, \tag{14}$$

$$\boldsymbol{v}(\Delta t) = \boldsymbol{v}(0) + \int_0^{\Delta t} \boldsymbol{v}(t) \times \boldsymbol{\omega}\ \mathrm{d}t. \tag{15}$$

As shown in Fig. 1, next we add the randomly oriented uncertainties $\Delta\boldsymbol{r}$, and $\Delta\boldsymbol{v}$ to $\boldsymbol{r}(\Delta t)$, and $\boldsymbol{v}(\Delta t)$, the magnitude of which is given by Eq. (7):

$$\boldsymbol{r}'(\Delta t) = \boldsymbol{r}(\Delta t) + \Delta\boldsymbol{r}, \tag{16}$$

$$\boldsymbol{v}'(\Delta t) = \boldsymbol{v}(\Delta t) + \Delta\boldsymbol{v}. \tag{17}$$

This procedure is repeated until the time $t$ reaches $\tau_c \equiv 2\pi/\omega$, the cyclotron period.

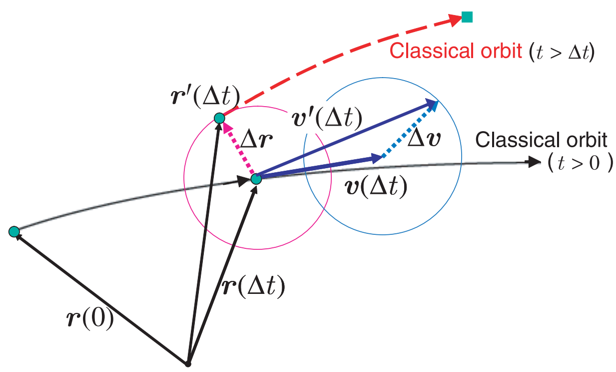


Fig. 1 Semiclassical model for quantum motion. Particle initially at $r(0)$ *classically* moves to $\boldsymbol{r}(\Delta t)$ with a velocity $\boldsymbol{v}(\Delta t)$ at $t = \Delta t - 0$. At this time, it suffers quantum mechanical deviations in position, $\Delta\boldsymbol{r}$, and in velocity, $\Delta\boldsymbol{v}$. The particle is at $\boldsymbol{r}'(\Delta t)$ with a velocity $\boldsymbol{v}'(\Delta t)$ at $t = \Delta t + 0$.

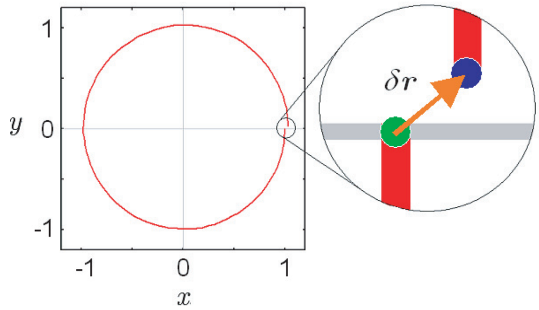


Fig. 2 Deviation of cyclotron motion, $\delta \boldsymbol{r} \equiv \boldsymbol{r}(\tau_c) - r(0)$, due to uncertainty in one gyration for a given characteristic length $L = v_0 \Delta t$. Lengths are normalized by the cyclotron radius $\rho = mv_0/qB$.

Figure 2 shows the particle's trajectory during one cyclotron period, in which a deviation $\delta \boldsymbol{r} \equiv \boldsymbol{r}(\tau_c) - r(0)$ from the classical motion is seen.

From many Monte Carlo calculations of such deviations (in this study, typically $N_{\mathrm{MC}} \sim 10^4$ turns out to be enough for convergence), the diffusion coefficient

$$D = \frac{\left\langle (\delta \boldsymbol{r})^2 \right\rangle}{\tau_c} \tag{18}$$

will be obtained for a particular choice of the characteristic length $L$, where $\langle \cdot \rangle$ stands for the ensemble average,

$$\left\langle X^2 \right\rangle \equiv \frac{1}{N_{\mathrm{MC}}} \sum_{i=1}^{N_{\mathrm{MC}}} X_i^2.$$

Figure 3 shows the histogram of $\delta \boldsymbol{r}$ for $N_{\mathrm{MC}} = 5 \times 10^7$ Monte Carlo trials, which resembles the probability density function of a wavefunction in quantum mechanics.

In the following subsections we will choose the average interparticle separation $\Delta \ell \equiv n^{-1/3}$, and the magnetic length, $\ell_{\mathrm{B}} \equiv \sqrt{\hbar/qB}$, as the characteristic length $L$.

### 3.1 Case A: $L$ = interparticle separation, $\Delta \ell$

If we choose the characteristic length $L \equiv \Delta \ell$, where $\Delta \ell \equiv n^{-1/3}$ stands for the average interparticle separation, the time interval is $\Delta t = \Delta \ell / v_0$, and the uncertainty in energy is given as $\Delta E \sim m v_0 \Delta v$. Thus, from Eq. (7), we have

$$\Delta r \sim \Delta \ell, \text{ and } \Delta v \sim \frac{\hbar}{m \Delta \ell}. \tag{19}$$

The particle is a proton in typical fusion plasmas of $T$ = 1-100 keV, $n = 10^{19}$-$10^{21}$ m$^{-3}$, and $B$ = 1-10 Tesla. The initial particle speed $v_0$ is selected as the thermal speed $v_{\mathrm{th}} = \sqrt{2T/m}$. The above calculation for a fixed $T$, $n$, and $B$ is repeated $N_{\mathrm{MC}} = 10^4$ times.

Figure 4 shows the temperature and density dependence of the diffusion coefficient $D = D(T, n)$, which leads to a scaling of

$$D_{\text{Case A}} \sim 0.094 \left( \frac{T_{\mathrm{keV}}}{A} \right)^{0.50} \left( \frac{10^{20}}{n} \right)^{0.33} \ \mathrm{m^2/s} \tag{20}$$
$$\propto \sqrt{\frac{T}{m}} n^{-1/3},$$

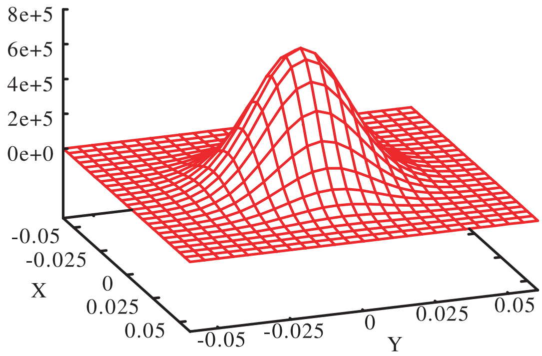


Fig. 3 Histogram for the deviation from the classical position, $\delta \boldsymbol{r} = (\delta x, \delta y)$, normalized by the cyclotron radius $\rho$ for $N_{\mathrm{MC}} = 5 \times 10^7$ Monte Carlo trials.

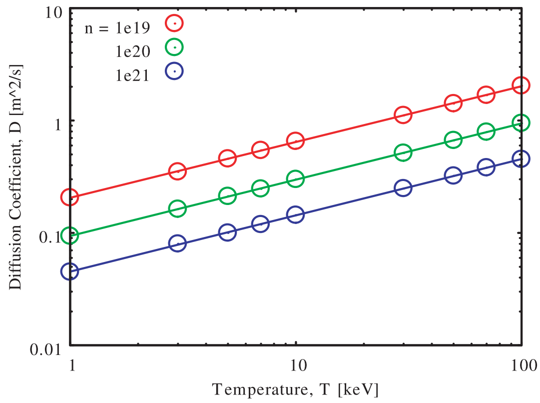


Fig. 4 Case A: Temperature $T$ and density $n$ dependence of diffusion coefficient $D$ [m$^2$/s] with fitting lines in the case of $L = \Delta \ell$, the interparticle separation.

where $A = m/m_p$ is the mass number, with $m_p$ being the proton mass. Interestingly, the diffusion coefficient $D$ does not depend on the magnetic field $B$, but on the particle mass $m^{-1/2}$. This is known as the isotope effect [5, 6].

### 3.2 Case B: $L$ = magnetic length, $\ell_{\mathrm{B}}$.

The uncertainties for Case B, $L = \ell_{\mathrm{B}}$, are

$$\Delta r \sim \ell_{\mathrm{B}}, \text{ and } \Delta v \sim \frac{\hbar}{m \ell_{\mathrm{B}}}. \tag{21}$$

In this case, the time interval is $\Delta t = \ell_{\mathrm{B}}/v_0$. The particle is a proton in typical fusion plasmas of $T$ = 1-100 keV, and $B$ = 1-10 Tesla. Note that the density $n$ does not enter into this case. Monte Carlo calculations are made, similar to those for Case A.

Figure 5 shows the temperature $T$ and magnetic field $B$ dependencies of the diffusion coefficient $D$, which leads to scaling of

$$D_{\text{Case B}} \sim 0.033 \left( \frac{T_{\mathrm{keV}}}{AB} \right)^{0.50} \ \mathrm{m^2/s} \propto \sqrt{\frac{T}{mB}}, \tag{22}$$

in which $\sqrt{T/m}$ scaling is the same as Eq. (21) for Case A.

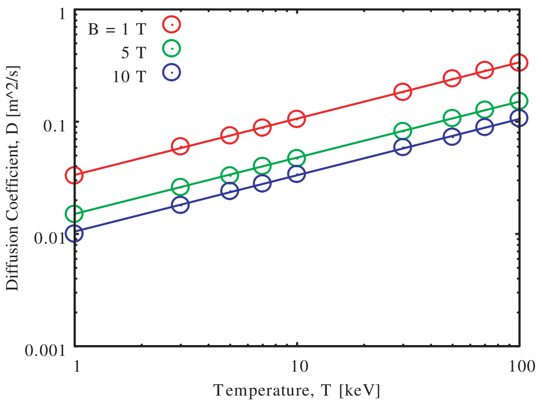


Fig. 5 Case B: Temperature $T$ and magnetic field $B$ dependence of diffusion coefficient $D$ [$\mathrm{m}^2$/s], with fitting lines in the case of $L = \ell_B$, the magnetic length.

## 4. Discussion

Case A, $L = \Delta\ell$, considers interactions among plasma particles, and Case B, $L = \ell_B$, considers the interaction of individual plasma particles with the magnetic field, i.e., the electrons moving in the external coils. Since both interactions should occur in magnetically confined fusion plasmas, we have the combined diffusion coefficient $D = D_{\mathrm{CaseA}} + D_{\mathrm{CaseB}}$ as

$$D \sim \left\{ \frac{0.094}{(n/10^{20})^{0.33}} + \frac{0.033}{\sqrt{B}} \right\} \sqrt{\frac{T_{\mathrm{keV}}}{A}}\ \mathrm{m}^2/\mathrm{s}. \qquad (23)$$

Table 1 summarizes the dependence of $D$ on plasma parameters such as $T$, $n$, and $B$. The parameters' ranges are $1 \le T \le 100\,\mathrm{keV}$, $1 \le B \le 10$ Tesla, and, if applicable, $10^{19} \le n \le 10^{21}\,\mathrm{m}^{-3}$. The ITER-89 dependence of $D$ in the table assumes $D \propto \rho^2/\tau_E$, where $\tau_E$ is the energy confinement time in the ITER-89 $L$-mode scaling law [7]. Note that in ITER-89 $L$-mode scaling, the temperature $T$ dependence is absent. Instead, it contains the heating power $P$, which should raise the temperature of the plasma; $T$ should be an increasing function of $P$. The diffusion coefficients $D$ from these models show a dependence on many parameters, such as the temperature $T$, mass $m$, density $n$, and magnetic field $B$, similar to that observed experimentally. Also, the proton diffusion coefficient is of the order of the anomalous diffusion. For a typical hydrogen fusion plasma (e.g., $T = 10\,\mathrm{keV}$, $n = 10^{20}\,\mathrm{m}^{-3}$, and $B = 3\,\mathrm{T}$), however, the value of the proton diffusion coefficients in our model are $D_{\mathrm{CaseA}} \sim 0.30\,\mathrm{m}^2/\mathrm{s}$ and $D_{\mathrm{CaseB}} \sim 0.08\,\mathrm{m}^2/\mathrm{s}$, both of which are one order smaller than the anomalous diffusion.

## 5. Summary

In current plasma physics based mainly on classical mechanics, quantum mechanical effects arise in the case of close encounters as a minor correction to the Coulomb logarithm $\ln\Lambda$. We have pointed out in this study that (i) for distant encounters in typical fusion plasmas of $T = 10\,\mathrm{keV}$ and $n = 10^{20}\,\mathrm{m}^{-3}$, the average potential energy $\langle U \rangle \sim 30\,\mathrm{meV}$ is as small as the uncertainty in energy $\Delta E \sim 40\,\mathrm{meV}$, and (ii) for a magnetic field $B = 3\,\mathrm{T}$, the spatial size of the wavefunction in the plane perpendicular to the magnetic field is as large as $\ell_B \sim 1.4 \times 10^{-8}\,\mathrm{m}$, which is much larger than the typical electron wavelength, $\lambda_e \sim 10^{-11}\,\mathrm{m}$.

The diffusion coefficients of our semiclassical model show a dependence on several plasma parameters, such as the temperature $T$ or the heating power input $P$, the mass $m$ or the isotope effect, the density $n$, and the magnetic field $B$, that are *qualitatively* similar to those observed experimentally, as well as having values larger than those for neoclassical diffusion.

In magnetically confined fusion plasmas, diffusion is governed by the banana particle motion due to the toroidicity of the magnetic field and the plasma current $I_p$, with which we have not dealt in this study. The diffusion model presented here is semiclassical, so we will need to solve Schrödinger's equation for exact analysis; this will be reported soon.

Table 1 Parameter dependence of $D$. The parameters' ranges are $1 \le T \le 100\,\mathrm{keV}$, $10^{19} \le n \le 10^{21}\,\mathrm{m}^{-3}$, and $1 \le B \le 10$ Tesla. The ITER-89 dependence of $D$ in the table assumes $D \sim \rho^2/\tau_E$. Values are for hydrogen plasmas.

| Model | $D \propto T^\alpha m^\beta n^\gamma B^\delta \cdots$ | Values |
|---|---|---|
| ITER-89 $L$-mode | $\sqrt{P/m}\, n^{-0.1} B^{-0.2}$ | $\sim 1\,\mathrm{m}^2/\mathrm{s}$ |
| Case A: $\Delta\ell = n^{-1/3}$ | $\sqrt{T/m}\, n^{-1/3} B^0$ | 0.1–2 |
| Case B: $\ell_B = \sqrt{\hbar/qB}$ | $\sqrt{T/m}\, n^0\, B^{-0.5}$ | 0.01–0.3 |
| Neo-classical theory | $\sqrt{m/T}\, n\, B^{-2}$ | $\sim 0.01$ |

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