MHD Stability in Flowing Plasmas: Connection between Fusion Plasma and Astrophysics Research

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Axisymmetric magneto-rotational instability (MRI) is studied in comparison with interchange instability (IntI) in a rotating cylindrical plasma. MRI is driven by the shear of plasma rotation, and the IntI by the density gradient with effective gravity due to the plasma rotation. The eigenmode equation for the MRI has the same form as that for the IntI. The local stability criterion is also summarized in a similar statement as "the spatial gradient of centrifugal force greater than the square of Aflvén frequency causes instability." However, the MRI is essentially different from the IntI because of the non-Hermitian property. The Keplerian rotation generates irregular singularity at the center of the disk, which yields a continuum of eigenvalues with non-orthogonal and square-integrable eigenfunctions.

Keywords: flowing plasma, magneto-rotational instability, interchange instability, Alfvén wave, irregular singularity

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1. Introduction

Flowing plasmas have attracted much attention in fusion plasma research as well as in astrophysics research. Especially in fusion plasma research, the plasma flow is attractive from the view point of physics as well as engineering. For example, it has been shown that a sheared plasma rotation (flow) stabilizes ballooning modes [1, 2] in fusion-oriented tokamak plasmas [3–6]. This fact itself is already attractive from the engineering point of view for designing a fusion reactor; if we utilize a plasma rotation, we may be able to achieve a higher beta value (the ratio of the plasma pressure to the magnetic pressure), which is desirable to build an efficient reactor. The physical mechanism of the stabilization has been explained as follows: the shear of the plasma rotation generates a new path for the energy transfer from an unstable ballooning mode to stable (continuum) modes [6]. This may be interesting from the physics point of view. The energy transfer was found by utilizing a complete set of basis functions which was obtained by regularizing the singular eigenfunctions belonging to the continuous spectrum. This regularization was accomplished by changing an eigenvalue problem yielding a continuous spectrum into a similar eigenvalue problem with a devised weight function which changes the continuum into discrete spectra and yields the same marginal stability and almost the same unstable eigenmodes as the original [7]. This shows that advanced techniques of mathematical physics are necessary to elucidate such interesting phenomena.

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Another example is the stability analysis of resistive-wall modes [8–11] which takes account of plasma rotation. The plasma rotation stabilizes the resistive-wall modes [9–11], which itself is attractive from the engineering point of view. The mechanism of the stabilization was explained as the coupling to the sound wave resonance [9, 10], which may be interesting from the physics point of view. This topic may also require advanced techniques of mathematical physics. The speed of plasma rotation in tokamaks is generally much lower than the Alfvén and sound speeds. Thus, the plasma rotation can be neglected in almost the entire region except for a thin layer around a rational or a resonant surface. Then the asymptotic matching technique can be applied.

A variety of interesting phenomena related to plasma flows other than listed above, such as drift wave turbulence and zonal flow [12], Kelvin-Helmholtz instability [13, 14] and others have been studied.

In astrophysics research, we also commonly find plasma flows such as accretion disks, jets, solar winds, etc. [15]. In the present paper, we concentrate on the accretion disk [16], which has a large-scale plasma flow around a massive central object such as a black hole. The accretion disk is an axisymmetric system, which means that the angular momentum of the plasma is conserved and the plasma cannot fall onto the massive central object. However, the observation shows rapid accretion, which cannot be explained by transport due to collisions and the turbulence of neutral fluids [16]. One of the most likely candidates to generate such an anomalous transport is turbulence due to the magneto-rotational instability (MRI) [17–19]. The MRI is driven by sheared plasma rotation, and can occur only in the presence of a magnetic field. Thus, it is usually studied by the magnetohydrodynamics (MHD) model, which is also frequently used for studying instabilities in fusion plasma research. We may expect some relationship between the MRI and instabilities known in fusion plasma research [20].

The focus of the present paper is therefore on studying the MRI and the instabilities in the fusion plasma on a common basis, as well as to discuss the similarities and differences: the keywords are the Alfvén wave and plasma flow. We chose the interchange instability (IntI) as an example of the instabilities in fusion plasmas. The IntI is driven by the density gradient with effective gravity due to plasma rotation. Below, the similarities and differences between the MRI and IntI will be shown.

This paper is organized as follows. In Sec. 2, the governing equations for the MRI and IntI are introduced. By applying the eigenmode approach to the governing equations, we will obtain the eigenvalue equation and the local stability criterion in Sec. 3. The local stability criteria for the MRI and IntI will be summarized in a single criterion. Section 4 shows the difference between the MRI and IntI by looking at the evolution equation for them. The IntI is governed by a Hermitian operator whereas the MRI is governed by a non-Hermitian operator. In Sec. 5, we will find the existence of an irregular singularity in the eigenmode equation in the case of the Keplerian rotation. This irregular singularity yields a continuum of eigenvalues. Concluding remarks are given in Sec. 6.

2. Governing Equations

Let us start with the simplest equilibrium in which magneto-rotational instability (MRI) and interchange instability (IntI) can occur. We assume a mass density $\rho = \rho(R)$, a constant pressure p, a constant magnetic field in the Z direction $\boldsymbol{B} = B\hat{Z}$ with $B = \text{const.}$, and a plasma rotation $\boldsymbol{v} = R\Omega(R)\hat{\theta}$ with an angular frequency $\Omega(R)$. The gravity of the massive central object is $-\rho g\hat{R}$ in the R direction. The cylindrical coordinate (R, θ, Z) is used (see Fig. 1), and $\hat{R}$, $\hat{\theta}$, $\hat{Z}$ denote unit vectors in the R, θ, Z directions, respectively. There is no electromagnetic force in the equilibrium.

The perturbation of the mass density ρ_1, the pressure p_1, the velocity $\boldsymbol{v}_1$ and the magnetic field $\boldsymbol{B}_1$ is assumed to be axisymmetric and incompressible as the simplest case, and the Z dependence of the perturbation is assumed to be $e^{\mathrm{i}kZ}$ as

$$\rho_1 = \tilde{\rho}(R)e^{\mathrm{i}kZ}, \tag{1}$$

$$p_1 = \tilde{p}(R)e^{\mathrm{i}kZ}, \tag{2}$$

$$\boldsymbol{v}_1 = \nabla[\phi(R)e^{\mathrm{i}kZ}] \times \nabla\theta + \tilde{v}_\theta(R)e^{\mathrm{i}kZ}\hat{\theta}, \tag{3}$$

$$\boldsymbol{B}_1 = \nabla[\psi(R)e^{\mathrm{i}kZ}] \times \nabla\theta + \tilde{B}_\theta(R)e^{\mathrm{i}kZ}\hat{\theta}. \tag{4}$$

By using the Boussinesq approximation [13], equations are

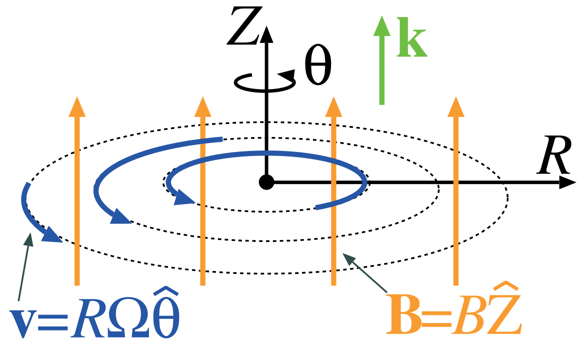

Fig. 1 The simplest equilibrium geometry in which MRI and IntI can occur.

written as

$$\frac{\partial\tilde{\rho}}{\partial t} = \mathrm{i}k\frac{\rho'(R)}{R}\phi, \tag{5}$$

$$\frac{\partial}{\partial t}\Delta^*\phi = \mathrm{i}k\left[-2R\Omega\tilde{v}_\theta + \frac{B}{\rho}\Delta^*\psi - \frac{R(R\Omega^2 + g)}{\rho}\tilde{\rho}\right], \tag{6}$$

$$\frac{\partial\tilde{v}_\theta}{\partial t} = \mathrm{i}k\left[(2\Omega + R\Omega'(R))\phi + \frac{B}{\rho}R\tilde{B}_\theta\right], \tag{7}$$

$$\frac{\partial\psi}{\partial t} = \mathrm{i}kB\phi, \tag{8}$$

$$\frac{\partial\tilde{B}_\theta}{\partial t} = \mathrm{i}k\left(B\tilde{v}_\theta - \Omega'(R)\psi\right), \tag{9}$$

where the prime denotes the derivative with respect to R, and Δ^* is defined as

$$\Delta^* := R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}\right) - k^2. \tag{10}$$

The variables in these equations are normalized by their typical values such as the system size R_0, the magnetic field B_0, the mass density ρ_0, and the Alfvén time $\tau_A = R_0/v_A$ with $v_A = B_0/\sqrt{\mu_0\rho_0}$. The pressure is normalized by the magnetic pressure. The "system size" will be discussed further when the Keplerian rotation is introduced in Sec. 5.

The vorticity equation (6) is used instead of the equation of motion. The Boussinesq approximation has been applied as mentioned above, and the density perturbation is included only through the gravity term including the centrifugal force term. The gravity term makes coupling between the vorticity equation and the equation of continuity. The shear Alfvén wave is described by the vorticity equation (6) with only its second term on the r.h.s. and the induction equation (8), or by Eqs. (7) and (9) with $\Omega = \Omega' = 0$. The MRI is described by Eqs. (6) with $\tilde{\rho} = 0$, (7), (8) and (9). The MRI cannot exist without the Alfvén wave, or $B = 0$. On the other hand, the IntI is described by the equation of continuity (5) and the vorticity equation (6) with only the third term on the r.h.s. The IntI can exist without the Alfvén wave, which is naturally understood since IntI can be unstable in neutral fluids. It is noted that the IntI in the present study is different from the IntI which we normally take into account in magnetically confined fusion plasmas. In a magnetically confined fusion plasma, IntI with $k = 0$ is most unstable, and thus we normaly assume $k = 0$. However, here we consider $k \neq 0$, or there is no rational surface, in order to compare with MRI which does not exist for $k = 0$. It is also noted that the IntI in the present study is axisymmetric and does not have a mode number in θ. The spatial structure of the perturbation is similar to the so-called sausage instability in a Z pinch [20], although the equilibrium magnetic field is purely in the Z direction in our study, not in the θ direction.

If we assume the time dependence of the perturbation as $e^{-i\omega t}$, the coupled equations (5) - (9) can be reduced to a single equation as

$$R\frac{d}{dR}\left(\frac{1}{R}\frac{d\phi}{dR}\right) - k^2 \times\left[1-\frac{2\Omega(2\Omega+R\Omega')D+4\Omega^2\omega_A^2}{D^2}+\frac{G_{\rm eff}}{D}\right]\phi=0, \tag{11}$$

where

$$D(\omega) := \omega^2 - \omega_A^2, \tag{12}$$

$$\omega_A^2 := k^2 v_A^2 := k^2\frac{B^2}{\rho}, \tag{13}$$

$$G_{\rm eff} := -\frac{\rho'(R\Omega^2 + g)}{\rho}. \tag{14}$$

The first term in the square braces of Eq. (11) corresponds to the Alfvén wave, the second to the MRI, and the third to the IntI. If we assume $\Omega \simeq 0$ and $\Omega' \neq 0$, which can only be appropriate locally, the second term in the square braces of Eq. (11) becomes $-2R\Omega\Omega'/D$. If we define $G_{\rm eff} := -R(\Omega^2)'$, this is exactly the same form as the third term for IntI.

The boundary conditions for the MRI may be controversial. Here we adopt $\phi = 0$ at $R = 0$ and at a finite radius of $R = R_a$. The radius R_a should be large enough, although the MHD model becomes invalid if R_a is too large. For the IntI, on the other hand, we have a finite plasma boundary at $R = R_a$, and $\phi = 0$ at $R = R_a$ for fixed boundary modes.

3. Local Stability Criterion

In this section, the local stability criterion is derived. Let us consider IntI first. If we approximate $R\frac{d}{dR}\left(\frac{1}{R}\frac{d\phi}{dR}\right)$ by $\frac{d^2\phi}{dR^2}$, which could be appropriate for large R, the eigenmode equation (11) becomes

$$\frac{\partial^2\phi}{\partial R^2} - k^2\left[1 + \frac{G_{\rm eff}}{D}\right]\phi = 0 \tag{15}$$

for the IntI. This equation has oscillatory solutions if

$$1 + \frac{G_{\rm eff}}{D} < 0, \tag{16}$$

and the solutions satisfy the boundary conditions. Then, we obtain a condition for instability as

$$G_{\rm eff} > \omega_A^2. \tag{17}$$

This criterion means that large enough effective gravity to bend the magnetic field causes instability.

Next, the eigenmode equation for the MRI can be similarly written as

$$\frac{d^2\phi}{dR^2} - k^2\left[1 - \frac{2\Omega(2\Omega + R\Omega')D + 4\Omega^2\omega_A^2}{D^2}\right]\phi = 0. \tag{18}$$

The condition for this equation to have oscillatory solutions is

$$1 - \frac{2\Omega(2\Omega + R\Omega')D + 4\Omega^2\omega_A^2}{D^2} < 0, \tag{19}$$

which is 4th order in ω. This inequality can be reduced as

$$-R(\Omega^2)' > \omega_A^2. \tag{20}$$

This means that a large enough rotation shear to bend the magnetic field causes instability. This is exactly the same form as the criterion for the IntI (17).

These local stability criteria for MRI and IntI can be summarized as follows:

$$R\frac{d(\rho\Omega^2)}{dR} > \rho\omega_A^2 \qquad \text{for instability.} \tag{21}$$

The l.h.s. of this inequality is the spatial gradient of the centrifugal force. Therefore, the MRI which is driven by the shear of plasma rotation and the IntI which is driven by the density gradient can be understood as similar instabilities.

4. Non-Hermitian Property of MRI

In the previous sections, the governing equations were described by using the mass density, vorticity, pressure and magnetic field since it may be convenient to see which term corresponds to which wave or instability. In the following, we will use a displacement vector $\boldsymbol{\xi}$ in order to combine those equations into a single equation. By using the displacement vector $\boldsymbol{\xi}$, the incompressible version of the linearized MHD equations can be written as [21]

$$\begin{aligned}\rho\frac{\partial^2\boldsymbol{\xi}}{\partial t^2} &+ 2\rho\boldsymbol{v}\cdot\nabla\frac{\partial\boldsymbol{\xi}}{\partial t} \\ &= [(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B} + (\nabla\times\boldsymbol{B})\times\boldsymbol{B}_1] \\ &\quad -\nabla p_1 + \boldsymbol{\xi}\cdot\nabla(\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v}) - \rho\boldsymbol{v}\cdot\nabla(\boldsymbol{v}\cdot\nabla\boldsymbol{\xi}),\end{aligned} \tag{22}$$

$$\nabla\cdot\boldsymbol{\xi} = 0, \tag{23}$$

where the perturbed magnetic field $\boldsymbol{B}_1$ is defined as

$$\boldsymbol{B}_1 := \nabla\times(\boldsymbol{\xi}\times\boldsymbol{B}). \tag{24}$$

It should be noted that the condition (23) does not necessarily mean incompressibility $\nabla\cdot\boldsymbol{v} = 0$. In the formulation of Eq. (22), the perturbed velocity field is defined as [21]

$$\boldsymbol{v}_1 := \frac{\partial\boldsymbol{\xi}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{v}. \tag{25}$$

If we assume $\nabla \cdot \boldsymbol{\xi} = 0$, it is shown that $\nabla \cdot \boldsymbol{v}_1$ vanishes when $\boldsymbol{\xi} \cdot \nabla(\nabla \cdot \boldsymbol{v}) = 0$. Thus, if the equilibrium velocity is divergence-free, $\nabla \cdot \boldsymbol{\xi} = 0$ means incompressibility $\nabla \cdot \boldsymbol{v}_1 = 0$.

By eliminating ξ_θ, ξ_Z and p_1 from Eqs. (22) and (23) and by using the Boussinesq approximation, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R}\frac{\mathrm{d}(R\xi_R)}{\mathrm{d}R}\right) - k^2 \times\left[1 - \frac{2\,\Omega(2\,\Omega + R\Omega')D + 4\,\Omega^2\omega_\mathrm{A}^2}{D^2} + \frac{G_\mathrm{eff}}{D}\right]\xi_R = 0, \quad (26)$$

which is equivalent with Eq. (11) by replacing ϕ with $R\xi_R$.

For $\Omega = \Omega' = 0$, we multiply Eq. (26) by $R\xi_R$ and integrate over R to obtain

$$(\omega^2 - \omega_\mathrm{A}^2)\int \mathrm{d}R\,\frac{1}{R}\left[\left|\frac{\mathrm{d}(R\xi_R)}{\mathrm{d}R}\right|^2 + k^2|R\xi_R|^2\right] + k^2 G_\mathrm{eff}\int \mathrm{d}R\,\frac{1}{R}|R\xi_R|^2 = 0. \quad (27)$$

This shows that the eigenvalue ω^2 for IntI is real. Therefore, the eigenvalue problem for the IntI possesses the Hermitian property. In fact, it can be confirmed by looking at the evolution equation corresponding to the eigenvalue equation (26). For $\Omega = \Omega' = 0$, we obtain

$$\left(\mathrm{i}\,\frac{\partial}{\partial t}\right)^2 L_\mathrm{IntI}\xi_R = (\omega_\mathrm{A}^2 L_\mathrm{IntI} + k^2 G_\mathrm{eff})\xi_R, \quad (28)$$

where

$$L_\mathrm{IntI} := \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}R\right) - k^2. \quad (29)$$

The operators on both sides of the above equation are Hermitian; therefore, the time evolution of ξ_R is governed by a Hermitian operator as a total.

Similarly, for $G_\mathrm{eff} = 0$, we obtain

$$-a(\omega^2 - \omega_\mathrm{A}^2)^2 + b(\omega^2 - \omega_\mathrm{A}^2) + c = 0, \quad (30)$$

where

$$a := \int \mathrm{d}R\,\frac{1}{R}\left[\left|\frac{\mathrm{d}(R\xi_R)}{\mathrm{d}R}\right|^2 + k^2|R\xi_R|^2\right], \quad (31)$$

$$b := 2\,k^2\int \mathrm{d}R\,\frac{1}{R}\Omega(2\,\Omega + R\Omega')|R\xi_R|^2, \quad (32)$$

$$c := 4\omega_\mathrm{A}^2 k^2\int \mathrm{d}R\,\frac{1}{R}\Omega^2|R\xi_R|^2. \quad (33)$$

It is obvious that a and c are positive, although b can change sign. We obtain

$$\omega^2 - \omega_\mathrm{A}^2 = \frac{-b \pm \sqrt{b^2 + 4ac}}{-2a}, \quad (34)$$

which shows that ω^2 is always real. Therefore, the eigenvalue problem for the MRI also seems to possess the Hermitian property like IntI. However, in fact it does not have the Hermitian property, as will be seen in the following. The evolution equation for the MRI can be written as

$$\left(\mathrm{i}\,\frac{\partial}{\partial t}\right)^2\begin{pmatrix} L_1 & I \\ L_2 & 0 \end{pmatrix}\begin{pmatrix} \xi_R \\ \zeta_R \end{pmatrix} = \omega_\mathrm{A}^2\begin{pmatrix} -L_0 & 0 \\ 0 & I \end{pmatrix}\begin{pmatrix} \xi_R \\ \zeta_R \end{pmatrix}, \quad (35)$$

where

$$L_2 := \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}R\right) - k^2, \quad (36)$$

$$L_1 := -2\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}R\right) + k^2\left(2 + \frac{R(\Omega^2)' + 4\,\Omega^2}{\omega_\mathrm{A}^2}\right), \quad (37)$$

$$L_0 := \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}R\right) - k^2\left(1 + \frac{R(\Omega^2)'}{\omega_\mathrm{A}^2}\right). \quad (38)$$

The matrix operator on the r.h.s. of Eq. (35) is Hermitian, but the matrix operator on the l.h.s. is not. Thus, the system is governed by a non-Hermitian operator as a total. Then, if we replace $\partial/\partial t$ by $-\mathrm{i}\,\omega$, ω^2 is generally complex. However, we saw that ω^2 was real for MRI. Therefore, the eigenvalue problem of MRI belongs to a special class among non-Hermitian systems.

5. Irregular Singularity and Continuum of Eigenvalues

In this section, we concentrate only on the MRI [22]. If we assume the Keplerian rotation $\Omega^2 = R^{-3}$, the eigenmode equation (26) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}R}\left(\frac{1}{R}\frac{\mathrm{d}(R\xi_R)}{\mathrm{d}R}\right) - k^2\left[1 - \frac{\omega^2 + 3\omega_\mathrm{A}^2}{(\omega^2 - \omega_\mathrm{A}^2)^2}\frac{1}{R^3}\right]\xi_R = 0. \quad (39)$$

It is noted that the Keplerian rotation is expressed as $\Omega^2 = GM/R^3$ in the physical units, where G is the gravitational constant and M is the mass of the central object. If we normalize this representation of the Keplerian rotation by using a scale length R_0, we obtain $\Omega^2 = R^{-3}$ with $\tau_\mathrm{A}^2 GM/R_0^3 = GM/v_\mathrm{A}^2 R_0 = 1$. This relationship determines the scale length R_0 as $R_0 = GM/v_\mathrm{A}^2$. The mass of the sun is about 2×10^{30} kg and the diameter of the largest accretion disk detected to date is about 2×10^4AU or 3×10^{15} m [23]. For a considerably wide range of parameters, the scale length R_0 may be much greater than the size of the accretion disks [22]. Assuming a sun-like star of one solar mass, the approximation $R \ll 1$, which will be used in the following analysis, does not necessarily mean that we concentrate only on the region very near the massive central object.

By introducing $\varphi = R^{1/2}\xi_R$, this equation can be cast in the form of the Schrödinger equation as

$$-\frac{d^2\varphi}{dR^2}+V(R;\omega^2)\varphi=0, \tag{40}$$

$$V(R;\omega^2):=\frac{3}{4R^2}+k^2\left[1-\frac{\omega^2+3\,\omega_A^2}{(\omega^2-\omega_A^2)^2}\frac{1}{R^3}\right]. \tag{41}$$

We see that $R = 0$ is an irregular singularity. Of course, the radius $R = 0$ is a mathematical artifact. We have to consider a finite-radius inner boundary for the idealized model of the disk dynamics. However, the existence of a singularity at $R = 0$ implies that the radial structures of the modes are highly sensitive to the boundary condition to be imposed near the axis. In fact, the inner boundary has been set at a finite R position near the origin in the previous studies. Then the resulting eigenfunctions localize around the inner boundary [24, 25], since the potential V becomes deeper as it approaches the inner boundary from the larger R side, and V becomes positive infinite abruptly at the boundary. It means that the location of the localization of the eigenfunctions changes if we change the position of the inner boundary. Therefore, we are motivated to analyze the effect of the singularity to understand the structure of modes near $R = 0$. Even if the singularity is removed by changing the model equation appropriately, the eigenmodes should inherit the characteristics of those with singularity.

Let us analyze the eigenmodes by assuming $R \ll 1$. Then the R^{-3} term becomes dominant, and we can approximate the eigenmode equation as

$$\frac{d^2\varphi}{dR^2}+\frac{\lambda}{R^3}\varphi=0, \tag{42}$$

where

$$\lambda:=k^2\frac{\omega^2+3\,\omega_A^2}{(\omega^2-\omega_A^2)^2}. \tag{43}$$

This equation can be solved for positive λ as

$$\varphi=C_1R^{1/2}J_1\left(2\sqrt{\frac{\lambda}{R}}\right)+C_2R^{1/2}Y_1\left(2\sqrt{\frac{\lambda}{R}}\right), \tag{44}$$

where J_1 and Y_1 are the first-order Bessel functions of the first and second kind, respectively, and C_1 and C_2 are arbitrary constants. As R becomes smaller, ξ_R oscillates more rapidly. The two independent solutions $\xi_R = \varphi/R^{1/2}$ are plotted in Fig. 2. The oscillation of φ is so rapid around $R = 0$ that the resolution is not sufficient in the plots. Both solutions satisfy the boundary condition $\xi_R = 0$ at $R = 0$. Then, if we try to satisfy two boundary conditions, i.e., $\xi_R = 0$ at $R = 0$ and an outer boundary $R = R_a$, only the ratio between C_1 and C_2 can be determined. In other words, the boundary conditions can be satisfied for arbitrary $\lambda > 0$ or $\omega^2 > -3\omega_A^2$.

It is noted that the eigenfunctions are square-integrable [22]. Thus we have found a continuum of eigenvalues even for $\omega^2 < 0$. The eigenfunctions are non-orthogonal with each other. In the ideal MHD model, it is known that the continua can exist only for $\omega^2 > 0$ and that the corresponding "eigenfunctions" are orthogonal with each other and singular or non square-integrable [26, 27]. The irregular singularity yields these curious behaviors of the eigenvalues and eigenfunctions.

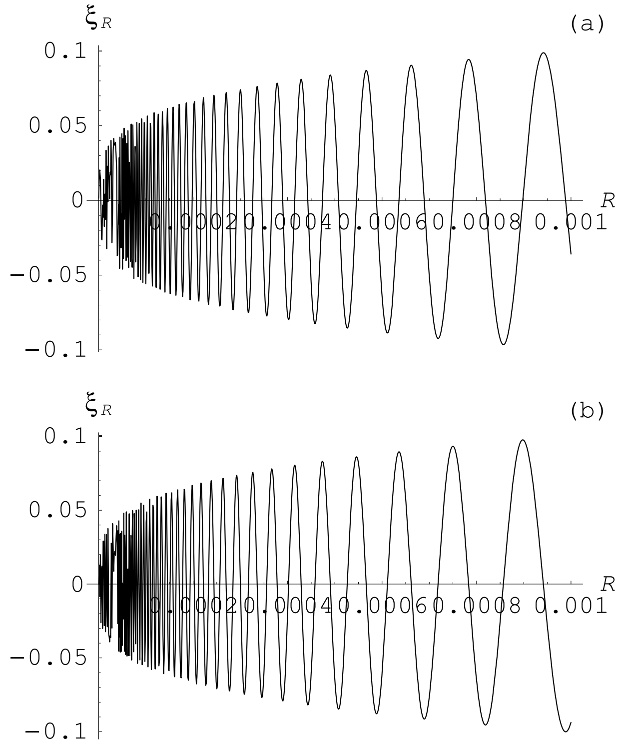

Fig. 2 The eigenfunctions $\xi_R = R^{-1/2}\varphi$ for $\lambda = 1$. (a): $J_1(2(\lambda/R)^{1/2})$, (b): $Y_1(2(\lambda/R)^{1/2})$. Both of them satisfy the boundary condition $\xi_R = 0$ at $R = 0$. The radial variation is rapid around $R = 0$ and becomes slower as R increases.

6. Conclusions

The eigenmode equations for the MRI and IntI have a similar form, and the local stability criterion can be summarized in a single statement such that the instability occurs if the spatial gradient of gravity including the centrifugal force is greater than the square of Alfvén frequency. The eigenvalues are real for both the MRI and IntI, but the MRI is essentially different from the IntI because of the non-Hermitian property of the governing equation. The irregular singularity at the center of the accretion disk in the case of the Keplerian rotation yields the continuum of eigenvalues with non-orthogonal and square-integrable eigenfunctions.

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