Stochastic Approach to Modeling Fluctuating Flow

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(Received 21 November 2005 / Accepted 3 February 2006)

In fluid equations describing edge plasma transport, the fluctuating flow causing anomalous transport is frequently interpreted as noise. The transport which is generated by the noise is represented as diffusion. In the present paper, the validity of the anomalous diffusion model of the fluctuating flow, i.e., $\overleftrightarrow{\Gamma}_{\mathrm{a}} = -\overleftrightarrow{\mathrm{D}}_{\mathrm{a}} \cdot \nabla \boldsymbol{v}$, is examined from the viewpoint of a stochastic approach to modeling, where $\boldsymbol{v}$ is a velocity field and $\overleftrightarrow{\mathrm{D}}_{\mathrm{a}}$ is a tensor of an anomalous diffusion coefficient. The examination is carried out on the presupposition that the validity of the diffusion model itself is not strongly related to details of the edge plasma. If the diffusion model is derived directly from the fundamental properties of the fluctuating flow, then the model is understood to be not merely an approximate description of the anomalous transport but to be inherent in the transport. However, it is found that because the noise given from the fluctuating flow is essentially bounded, the transport modeling is not justified.

Keywords: transport modeling, anomalous diffusion, edge plasma, stochastic phenomenon, Langevin equation, stochastic analysis

DOI: 10.1585/pfr.1.012

1. Introduction

In studies of edge plasma transport, a simplified fluid model is frequently employed if the collisionality of the plasma particles is strong [1]. The fluid model is based on simplified Braginskii's fluid equations neglecting the electric field and plasma current; i.e., the simple neutral plasma is assumed, where the fluid equations of the density, momentum, and energy are written in the Fokker-Planck form [1–4]. From the analogy of the Feynman-Kac (FK) formula [5], stochastic differential equations (or Langevin equations) are used to solve the fluid equations describing the edge plasma transport in a three-dimensional (3D) magnetic field line structure including an ergodic zone with magnetic islands [2–4]. This is because of the difficulty in realizing the partial differential operators along and across a field line, $\nabla_{\parallel}$ and $\nabla_{\perp}$, in the 3D field line structure. However, since the equation of the fluid motion, i.e., the Navier-Stokes (NS) -type equation, is nonlinear, the FK formula cannot be directly applied to it. Another stochastic approach which is based on Yasue's theorem [6, 7] is needed to understand the statistical properties of the transport, as seen in Sec. 2.

As observed in most experiments [1], the transport of edge plasma is large compared with the classical diffusion. In the fluid equations, Fick's law of diffusion, $\vec{\Gamma}_{\mathrm{a}} = -\overleftrightarrow{\mathrm{D}}_{\mathrm{a}} \cdot \nabla f$, is frequently employed to describe the anomalous transport generated by the fluctuating flow in the edge plasma [1–4, 8, 9], where a fluid quantity f expresses the density, momentum (or velocity), and energy (or temperature), and $\overleftrightarrow{\mathrm{D}}_{\mathrm{a}}$ is an anomalous diffusion coefficient. The present paper is devoted to, in particular, examining the validity of the diffusion model of the fluctuating flow in the equation of fluid motion, i.e., $\overleftrightarrow{\Gamma}_{\mathrm{a}} = -\overleftrightarrow{\mathrm{D}}_{\mathrm{a}} \cdot \nabla \boldsymbol{v}$, from the viewpoint of the stochastic approach to the modeling, where $\boldsymbol{v}$ denotes the velocity field. If the diffusion model is derived directly from fundamental properties of the fluctuating flow, then the model is understood to be not merely an approximate description of the anomalous transport but to be inherent in the transport. The examination is carried out without support of numerical simulations, and the validity is investigated under the following conditions: (i) a constant mass density, (ii) the incompressibility $\nabla \cdot \boldsymbol{v} = 0$, (iii) interactions between the plasma and neutrals or solid surfaces are neglected, (iv) a simple viscosity term describing the effect of the Coulomb collision, i.e., the simple representation of the classical diffusion: $-\nabla \cdot (\overleftrightarrow{\mathrm{D}}_{\mathrm{c}} \cdot \nabla \boldsymbol{v}) = -\nu \nabla^2 \boldsymbol{v}$ with a constant viscosity coefficient ν = constant, and (v) the simplified fluid has the periodic boundary in 3D space. Although the state of edge plasma under these conditions is far from the actual situation, the validity of the diffusion model itself is considered to be not strongly related to the details of edge plasma.

This paper is organized as follows. In Sec. 2, we examine the validity of the diffusion model of the fluctuating flow from the viewpoint of a Langevin equation which is the equation of the fluid particle motion corresponding to the NS equation. In Sec. 2.1, the stochastic approach checking the diffusion model is briefly introduced. In Sec. 2.2, it is found that if noise in the velocity field is essentially bounded, then the diffusion model is not justified. Finally, a summary is given in Sec. 3.

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2. Stochastic Approach to Fluctuating Flow

2.1 Navier-Stokes equation and stochastic process

We focus on the equation of fluid motion, i.e., the NS equation, in 3D space $\mathbf{R}^3$. By assuming the incompressibility $\nabla \cdot \boldsymbol{v} = 0$ and the constant mass density,

$$\left\{\frac{\partial}{\partial t} + \boldsymbol{v}(t,\boldsymbol{x}) \cdot \nabla - \nu\nabla^2\right\}\boldsymbol{v}(t,\boldsymbol{x}) = -\nabla\varphi(t,\boldsymbol{x}), \tag{1}$$

where the source/sink of momentum is assumed to be given as $\boldsymbol{S}_{\mathrm{m}} = -\nabla\varphi(t,\boldsymbol{x})$, φ is a scalar function, and $\boldsymbol{x}$ denotes an element of the set $\mathbf{R}^3$, i.e., $\boldsymbol{x} \in \mathbf{R}^3$. From Yasue's theorem [6, 7], the variation of the functional J_x gives the NS equation (1);

$$J_x = \int_{t_a}^{t_b} \mathrm{d}t\, \mathrm{E}\left[\frac{1}{2}\left|D\boldsymbol{X}_t\right|^2 - \varphi(t,\boldsymbol{X}_t)\right], \tag{2}$$

where $\mathrm{E}[\cdot]$ denotes the mathematical expectation. A volume-preserving diffusion process $\boldsymbol{X}_t$ having start and end points $\boldsymbol{X}_{t_a} = \boldsymbol{x}_a$ and $\boldsymbol{X}_{t_b} = \boldsymbol{x}_b$ is mean forward differentiable:

$$D\boldsymbol{X}_t := \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\left[\boldsymbol{X}_{t+\varepsilon} - \boldsymbol{X}_t\middle|\mathcal{P}_t^{X}\right] = \boldsymbol{v}(t,\boldsymbol{X}_t), \tag{3}$$

where the notation $\downarrow$ means "decreases to," and sometimes the limit in Eq. (3) may denote $\lim_{\varepsilon\to 0+}$ [10]. Here, $\mathrm{E}[\cdot|\mathcal{P}_t^X]$ denotes the conditional expectation with respect to $\mathcal{P}_t^X$, the σ-algebra $\mathcal{P}_t^X$ is generated by $\{\boldsymbol{X}_s; t_a \le s \le t\}$ [11], and the stochastic process $\boldsymbol{X}_t$ satisfies the Langevin equation:

$$\mathrm{d}\boldsymbol{X}_t = \boldsymbol{v}(t,\boldsymbol{X}_t)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}\boldsymbol{W}_t, \tag{4}$$

where $\boldsymbol{W}_t$ denotes a Wiener process [10]. Thus, the equation of motion of a fluid particle is given as the Langevin equation (4). The theorem implies that the stochasticity of the fluctuating flow can be understood through a stochastic process $\boldsymbol{Y}_t$ satisfying $D\boldsymbol{Y}_t = \boldsymbol{v}(t,\boldsymbol{Y}_t)$ and $\delta J_Y = 0$.

2.2 Statistics of fluctuating flow

In studies of edge plasma, the fluctuating flow causing anomalous transport is frequently interpreted as noise. We then introduce the equation of motion of a fluid particle having turbulent Lagrangian velocity $\tilde{\boldsymbol{u}}(t,\omega)$ in which randomness is inherent,

$$\mathrm{d}\boldsymbol{Y}_t = \tilde{\boldsymbol{u}}(t,\omega)\mathrm{d}t + \sqrt{2\nu}\mathrm{d}\boldsymbol{W}_t, \tag{5}$$

where the stochastic process $\boldsymbol{Y}_t : [t_a,t_b]\times\Omega \to \mathbf{R}^3$ is an Itô process, Ω is the ensemble of the fluid particles (or the sample space), ω is a typical element (or a sample point) of Ω, i.e., $\omega \in \Omega$, $\mathbf{R}^3$ is the 3D Euclidean space, and $\boldsymbol{x} \in \mathbf{R}^3$; see also Ref. [5]. Here, the noise $\tilde{\boldsymbol{u}}(t,\omega)$ is assumed to satisfy

$$\mathrm{E}\left[\tilde{\boldsymbol{u}}(t,\omega)\middle|\mathcal{P}_t^Y\right] = \boldsymbol{u}(t,\boldsymbol{Y}_t), \tag{6}$$

where $\boldsymbol{u}(t,\boldsymbol{x}) = (u^1(t,\boldsymbol{x}), u^2(t,\boldsymbol{x}), u^3(t,\boldsymbol{x}))$, and $\nabla\cdot\boldsymbol{u} = 0$.

The fluctuating flow is understood to be noise because of the lack of accurate information regarding the motions of the fluid particles, e.g., their initial conditions. If the fluctuating flow is a solution of the equation of fluid motion, then it is natural that the noise $\tilde{\boldsymbol{u}}(t,\omega)$ given from the fluctuating flow is essentially bounded, where there is a positive constant U satisfying the condition $|\tilde{\boldsymbol{u}}(t,\omega)| \le U$ for almost all $t \in [t_a,t_b]$ and almost all $\omega \in \Omega$. One may consider that the fluctuating flow is generated by perturbed electric/magnetic fields [1, 8, 9]. Because the perturbed fields satisfy the Maxwell equations, it is also natural that the turbulent velocity $\tilde{\boldsymbol{u}}(t,\omega)$ in this case is essentially bounded.

Since the velocity $\tilde{\boldsymbol{u}}$ is essentially bounded as $|\tilde{\boldsymbol{u}}(t,\omega)| \le U$, the velocity $\tilde{\boldsymbol{u}}(t,\omega)$ satisfies

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\left[\varepsilon^2\tilde{u}^i(t,\omega)\tilde{u}^j(t,\omega)g(t,\boldsymbol{Y}_t)\middle|\mathcal{P}_t^Y\right]\\
&\quad= \lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{E}\Big[(\tilde{u}_+^i - \tilde{u}_-^i)(\tilde{u}_+^j - \tilde{u}_-^j)\\
&\qquad\quad \left\{g^+(t,\boldsymbol{Y}_t) - g^-(t,\boldsymbol{Y}_t)\right\}\Big|\mathcal{P}_t^Y\Big]\\
&\quad= 0,
\end{aligned} \tag{7}$$

where ε is a time interval, $g(t,\boldsymbol{x})$ is a smooth function satisfying $|g(t,\boldsymbol{x})| < \infty$, $g^+ = \max\{g,0\}$, $g^- = \max\{-g,0\}$, $\tilde{\boldsymbol{u}} = (\tilde{u}^1,\tilde{u}^2,\tilde{u}^3)$, $\tilde{u}_+^i = \max\{\tilde{u}^i,0\}$, $\tilde{u}_-^i = \max\{-\tilde{u}^i,0\}$, and

$$0 \le \lim_{\varepsilon\downarrow 0}\varepsilon\mathrm{E}\left[\tilde{u}_\pm^i\tilde{u}_\pm^j g^\pm(t,\boldsymbol{Y}_t)\middle|\mathcal{P}_t^Y\right] \le \lim_{\varepsilon\downarrow 0}\varepsilon U^2 g^\pm(t,\boldsymbol{Y}_t) = 0. \tag{8}$$

In the same manner, we have the following for the case of $n \ge 3$:

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\left[\varepsilon^n\tilde{u}^{i_1}(t,\omega)\tilde{u}^{i_2}(t,\omega)\cdots\tilde{u}^{i_n}(t,\omega)g(t,\boldsymbol{Y}_t)\middle|\mathcal{P}_t^Y\right] = 0, \tag{9}$$

where $i_k = 1, 2, 3$ for $k = 1, 2, \ldots, n$. The velocity $\tilde{\boldsymbol{u}}(t,\omega)$ also satisfies

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\left[\varepsilon\tilde{u}^i(t,\omega)\left\{W_{t+\varepsilon}^j - W_t^j\right\}g(t,\boldsymbol{Y}_t)\middle|\mathcal{P}_t^Y\right] = 0, \tag{10}$$

where $\boldsymbol{W}_t = (W_t^1, W_t^2, W_t^3)$. In the same manner, we have the following for the case of $n \ge 1$ and $m \ge 1$:

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Big[\varepsilon^n\tilde{u}^{i_1}(t,\omega)\tilde{u}^{i_2}(t,\omega)\cdots\tilde{u}^{i_n}(t,\omega)\\
&\qquad\left\{W_{t+\varepsilon}^{j_1} - W_t^{j_1}\right\}\left\{W_{t+\varepsilon}^{j_2} - W_t^{j_2}\right\}\\
&\qquad\cdots\left\{W_{t+\varepsilon}^{j_m} - W_t^{j_m}\right\}g(t,\boldsymbol{Y}_t)\Big|\mathcal{P}_t^Y\Big] = 0.
\end{aligned} \tag{11}$$

Note that only the condition $|\tilde{\boldsymbol{u}}| \le U$ is used in Eqs. (7)-(11) and the detailed statistical properties of $\tilde{\boldsymbol{u}}$ are not required.

As a result, we have the following: for a smooth function $g(t,\boldsymbol{x})$

$$
\begin{aligned}
&Dg(t,\boldsymbol{Y}_t)\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Big[g(t+\varepsilon,\boldsymbol{Y}_{t+\varepsilon})-g(t,\boldsymbol{Y}_t)\Big|\mathcal{P}^Y_t\Big]\\
&= \lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Bigg[\varepsilon\frac{\partial g}{\partial t}(t,\boldsymbol{Y}_t)+\Big\{\boldsymbol{Y}_{t+\varepsilon}-\boldsymbol{Y}_t\Big\}\cdot\nabla g(t,\boldsymbol{Y}_t)\\
&\quad +\frac{1}{2}\sum_{i,j}\Big\{Y^i_{t+\varepsilon}-Y^i_t\Big\}\Big\{Y^j_{t+\varepsilon}-Y^j_t\Big\}\frac{\partial^2 g}{\partial x^i\partial x^j}(t,\boldsymbol{Y}_t)+\cdots\Bigg|\mathcal{P}^Y_t\Bigg]\\
&= \left(\frac{\partial}{\partial t}+\boldsymbol{u}\cdot\nabla+\nu\nabla^2\right)g(t,\boldsymbol{Y}_t),
\end{aligned}
\tag{12}
$$

where the displacement of $\boldsymbol{Y}_t=(Y^1_t,Y^2_t,Y^3_t)$ is given as

$$
\Big\{Y^i_{t+\varepsilon}-Y^i_t\Big\}=\varepsilon\tilde{u}^i(t,\omega)+\sqrt{2\nu}\Big\{W^i_{t+\varepsilon}-W^i_t\Big\},
\tag{13}
$$

the second term on the right-hand side of Eq. (12) is derived from

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Bigg[\Big\{Y^i_{t+\varepsilon}-Y^i_t\Big\}\frac{\partial g}{\partial x^i}(t,\boldsymbol{Y}_t)\Bigg|\mathcal{P}^Y_t\Bigg]\\
&\quad= \mathrm{E}\Big[\tilde{u}^i(t,\omega)\Big|\mathcal{P}^Y_t\Big]\frac{\partial g}{\partial x^i}(t,\boldsymbol{Y}_t)\\
&\quad= u^i(t,\boldsymbol{Y}_t)\frac{\partial g}{\partial x^i}(t,\boldsymbol{Y}_t),
\end{aligned}
\tag{14}
$$

and the following are obtained: for the case of $n=1$ or $n\geq 3$

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Bigg[\Big\{W^{i_1}_{t+\varepsilon}-W^{i_1}_t\Big\}\Big\{W^{i_2}_{t+\varepsilon}-W^{i_2}_t\Big\}\cdots\Big\{W^{i_n}_{t+\varepsilon}-W^{i_n}_t\Big\}\\
&\qquad\frac{\partial^n g}{\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_n}}(t,\boldsymbol{Y}_t)\Bigg|\mathcal{P}^Y_t\Bigg]=0,
\end{aligned}
\tag{15}
$$

and for $n=2$

$$
\begin{aligned}
&\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathrm{E}\Bigg[\Big\{W^i_{t+\varepsilon}-W^i_t\Big\}\Big\{W^j_{t+\varepsilon}-W^j_t\Big\}\frac{\partial^2 g}{\partial x^i\partial x^j}(t,\boldsymbol{Y}_t)\Bigg|\mathcal{P}^Y_t\Bigg]\\
&\quad= \delta^{ij}\frac{\partial^2 g}{\partial x^i\partial x^j}(t,\boldsymbol{Y}_t);
\end{aligned}
\tag{16}
$$

see also Ref. [5]. Note that the terms having the nth order derivatives $\partial^n g/\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_n}$ with $n\geq 3$ vanish in Eq. (12). Consequently, we see the following:

$$
\begin{aligned}
0 &= \delta J_Y\\
&= \int_{t_a}^{t_b}\mathrm{d}t\,\mathrm{E}\Big[\Big(D\delta\boldsymbol{Y}_t\Big)\cdot\Big(D\boldsymbol{Y}_t\Big)-\delta\boldsymbol{Y}_t\cdot\nabla\varphi(t,\boldsymbol{Y}_t)\Big]\\
&= \int_{t_a}^{t_b}\mathrm{d}t\int_{\mathcal{M}}\mathrm{d}V(\boldsymbol{x})\,\mu(t,\boldsymbol{x})\\
&\qquad\Big\{\Big(A\boldsymbol{h}(t,\boldsymbol{x})\Big)\cdot\boldsymbol{u}(t,\boldsymbol{x})-\boldsymbol{h}(t,\boldsymbol{x})\cdot\nabla\varphi(t,\boldsymbol{x})\Big\}\\
&= -\int_{t_a}^{t_b}\mathrm{d}t\int_{\mathcal{M}}\mathrm{d}V(\boldsymbol{x})\,\mu(t,\boldsymbol{x})\\
&\qquad\boldsymbol{h}(t,\boldsymbol{x})\cdot\Big\{A_*\boldsymbol{u}(t,\boldsymbol{x})+\nabla\varphi(t,\boldsymbol{x})\Big\},
\end{aligned}
\tag{17}
$$

where $\mathcal{M}$ is a region in which the fluid exists, $\mathrm{d}V(\boldsymbol{x})$ is the volume element, the probability density of the process $\boldsymbol{Y}_t$, i.e., $\mu(t,\boldsymbol{x})$, is constant because of the constant mass density, $\boldsymbol{h}(t,\boldsymbol{x})=\delta\boldsymbol{Y}_t$ is an arbitrary smooth function satisfying $\boldsymbol{h}(t_a,\boldsymbol{x}_a)=\boldsymbol{h}(t_b,\boldsymbol{x}_b)=0$, and the differential operators A and A_* are given as $A=\partial/\partial t+\boldsymbol{u}\cdot\nabla+\nu\nabla^2$ and $A_*=\partial/\partial t+\boldsymbol{u}\cdot\nabla-\nu\nabla^2$, respectively. Thus, we have

$$
\left\{\frac{\partial}{\partial t}+\boldsymbol{u}(t,\boldsymbol{x})\cdot\nabla-\nu\nabla^2\right\}\boldsymbol{u}(t,\boldsymbol{x})=-\nabla\varphi(t,\boldsymbol{x}).
\tag{18}
$$

Therefore the turbulent velocity $\tilde{\boldsymbol{u}}(t,\omega)$ cannot affect the viscosity term in the NS equation if $\tilde{\boldsymbol{u}}(t,\omega)$ is essentially bounded.

3. Summary

In the present paper, we have considered the question whether the diffusion model of the fluctuating flow is inherent in anomalous transport, from the viewpoint of the stochastic approach to modeling the edge plasma in the NS-type equation. The examination is carried out based on the presupposition that the validity of the anomalous diffusion model itself is not strongly related to the details of the edge plasma.

Because the NS-type equation is nonlinear, the well-known approach to the modeling shown in, for example, Refs. [12–14] cannot explain the anomalous diffusion in the equation. We should note that this approach indispensably employs the linearity of a fluid quantity.

Thus, we have introduced the stochastic approach based on Yasue's theorem [6, 7]. The fluctuating flow is understood to be bounded noise because the nonlinearity of the fluctuating flow causes a lack of the accurate information and the fluctuating flow itself is bounded, as discussed in Sec. 2. We see that the diffusion model is based on the assumption of an unboundedly fluctuating flow. It is found that because the noise is essentially bounded, the transport modeling is not justified.

In future work, the stochastic approach will be developed into a useful method of realistically modeling edge plasma.

Acknowledgments

This work was performed with the support and under the auspices of the NIFS Collaborative Research Program NIFS04KLDD001 and NIFS05ULDD101.

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