WKB Analysis of Axisymmetric Magneto-Rotational Instability in a Thin Accretion Disk

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The temporal behavior of axisymmetric magneto-rotational instability in a thin accretion disk is analyzed via the Wentzel-Kramars-Brillouin (WKB) method. The height of the thin disk is used as a small parameter. It is found that the oscillation of the envelope of the mode accelerates with time because of the density distribution in the direction of the disk height.

Keywords: magneto-rotational instability, accretion disk, WKB analysis

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Magneto-rotational instability (MRI) [1, 2] in accretion disks [3] has attracted much attention in astrophysics research since Balbus and Hawley pointed it out as a candidate for explaining the "anomalous" angular momentum transport in accretion disks [4]. In Ref. [4], the MRI was studied by local analysis which assumes a sinusoidal wave in the radial and height directions of the accretion disk. The global mode was discussed in, for example, Ref. [5]; however, they also adopted a sinusoidal wave in one of the radial and height directions. It is also noted that most of the MRI studies, except for the nonlinear numerical studies, are based on the eigenvalue approach. In this paper, we analyze the temporal behavior of the axisymmetric MRI in a thin accretion disk as an initial-value problem. The Wentzel-Kramars-Brillouin (WKB) method is applied and the sinusoidal wave is assumed in neither the radial nor height directions. The height of the thin disk is utilized as a small parameter in the WKB analysis.

In the present study, we adopt the ideal magnetohydrodynamics (MHD) model. Before analyzing the stability, we briefly mention the equilibrium of the accretion disk. Here we assume a simple geometry with the velocity field $\boldsymbol{v} = R\Omega(R)\hat{\varphi}$ and the magnetic field $\boldsymbol{B} = B\hat{Z}$, where $\Omega(R)$ is an angular rotation frequency, B is a constant, $\hat{\varphi}$ and $\hat{Z}$ are the unit vectors in the directions φ and Z in the cylindrical coordinates (R, φ, Z), respectively. Then, from the force-balance equation

$$\rho\boldsymbol{v}\cdot\nabla\boldsymbol{v} = (\nabla\times\boldsymbol{B})\times\boldsymbol{B} - \nabla p - \rho\nabla\Phi, \tag{1}$$

we obtain the mass density ρ and the angular rotation frequency Ω as

$$\rho = \rho_0(R)\exp\left[\frac{GM}{c_s^2}\left\{(R^2+Z^2)^{-1/2} - R^{-1}\right\}\right], \tag{2}$$

$$\Omega^2 = \frac{1}{R}\left[c_s^2\frac{\partial\ln\rho_0}{\partial R} + \frac{GM}{R^2}\right], \tag{3}$$

where p is the pressure, Φ is the gravitational potential, ρ_0 is an integration constant, $c_s := \sqrt{p/\rho}$ is the isothermal sound speed assumed to be constant, G is the gravitational constant, and M is the mass of the central object. The self-gravity is neglected. If we drop $\partial(\ln\rho_0)/\partial R$, we obtain the Keplerian rotation $\Omega^2 = GMR^{-3}$, for which Eq. (2) can be rewritten as

$$\rho = \rho_0(R)\exp(-Z^2/H^2), \tag{4}$$

$$H := \sqrt{2}c_s/\Omega \tag{5}$$

in the limit of $R \gg Z$. The scale height H is much smaller than typical radius R when the plasma rotation velocity $R\Omega$ is much larger than the sound speed c_s.

In the stability analysis, the mode structure we are considering is shown in Figure 1. It has a very short wave length in the Z direction, and has also an envelope in Z whose scale length is comparable to the scale height of the accretion disk. In order to express such a mode structure,

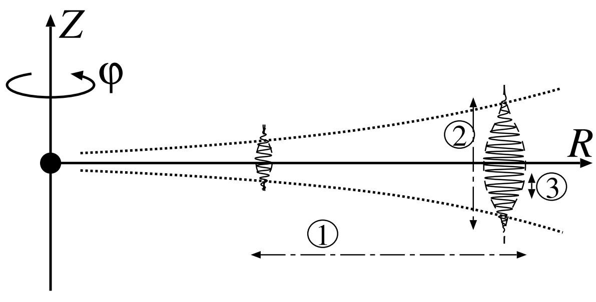

Fig. 1 The mode structure with three different scale lengths.

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we introduce the following form for perturbed quantities $\tilde{\boldsymbol{Q}}$ such as $\tilde{\rho}$, $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{B}}$:

$$\tilde{\boldsymbol{Q}}(R,Z,t) = \sum_{j=0}^{\infty} H_0^j \tilde{\boldsymbol{Q}}^{(j)}(R,Z,t) \times \exp\left[\mathrm{i}\, H_0^{-2} S_0(Z,t) + \mathrm{i}\, H_0^{-1} S_1(R,Z,t)\right], \tag{6}$$

where $H_0 := \sqrt{2}c_\mathrm{s}/(R_0\Omega_0)$ is a small parameter and Ω_0 is the angular rotation frequency at a representative radius $R = R_0$. Then $(1/\tilde{\boldsymbol{Q}})\partial\tilde{\boldsymbol{Q}}/\partial Z \sim k_Z \sim O(H_0^{-2})$. Also, $\Omega \sim O(H_0^{-1})$ from Eq. (5) when $c_\mathrm{s} \sim O(1)$. For the MRI to occur, the plasma rotation frequency should not be significantly smaller than the Alfvén frequency [4]; we should have $\Omega \sim k_Z v_\mathrm{A}$ where $v_\mathrm{A} := B/\sqrt{\rho}$ is the Alfvén velocity. Then v_A or B should be $O(H_0)$. Finally, note that the equilibrium density gradient in the Z direction is treated as $O(H_0^{-1})$.

Substituting Eq. (6) into the linearized MHD equations and using the ordering described above, we obtain, in the lowest order $O(H_0^{-2})$,

$$\omega_0^5(\omega_0^2 - c_\mathrm{s}^2 k_{Z0}^2) = 0, \tag{7}$$

where $\omega_0 := \partial S_0/\partial t$ and $k_{Z0} := \partial S_0/\partial Z$. Then we obtain $\omega_0 = 0$ and $\omega_0^2 - c_\mathrm{s}^2 k_{Z0}^2 = 0$. The latter gives the sound wave which propagates in the Z direction. Here we are not interested in that branch; we take $\omega_0 = 0$. Then $S_0 = S_0(Z)$. The corresponding eigenvector is given by $\tilde{\rho}^{(0)} = \tilde{v}_Z^{(0)} = 0$ and the other quantities $\tilde{v}_R^{(0)}$, $\tilde{v}_\varphi^{(0)}$, $\tilde{B}_R^{(0)}$, $\tilde{B}_\varphi^{(0)}$, and $\tilde{B}_Z^{(0)}$ are arbitrary.

The equations in the next order $O(H_0^{-1})$ give the following dispersion relation

$$\omega_1^2 = k_{Z0}^2\bar{v}_\mathrm{A}^2 + \bar{\Omega}(2\bar{\Omega} + R\bar{\Omega}') \pm \sqrt{\bar{\Omega}^2\left[4k_{Z0}^2\bar{v}_\mathrm{A}^2 + (2\bar{\Omega} + R\bar{\Omega}')^2\right]}, \tag{8}$$

where $\omega_1 := \partial S_1/\partial t$, $k_R := \partial S_1/\partial R$, $\bar{\Omega} := H_0\Omega$ and $\bar{v}_\mathrm{A} := v_\mathrm{A}/H_0$. The prime denotes the derivative with respect to R. This is basically the same with the dispersion relation obtained by the conventional analysis [4]. The minus-sign branch yields instability. The eigenvector yields $\tilde{B}_Z^{(0)} = 0$, and $\tilde{v}_R^{(0)}$, $\tilde{v}_\varphi^{(0)}$ and $\tilde{B}_\varphi^{(0)}$ can be represented in terms of $\tilde{B}_R^{(0)}$.

The equations in $O(1)$ can be summarized in a single equation for $\tilde{B}_R^{(0)}$ by utilizing the eigenvector obtained in the previous order as

$$\frac{1}{\tilde{B}_R^{(0)}}\frac{\partial\tilde{B}_R^{(0)}}{\partial t} = \mathrm{i}\,\frac{k_{Z1}}{\omega_1}\bar{v}_\mathrm{A}^2 k_{Z0} \times\left[1 \pm 2|\bar{\Omega}|\left\{4\bar{v}_\mathrm{A}^2 k_{Z0}^2 + (2\bar{\Omega} + R\bar{\Omega}')^2\right\}^{-1/2}\right], \tag{9}$$

where $k_{Z1} := \partial S_1/\partial Z$. Since $\partial S_1/\partial t = \omega_1(R,Z;k_{Z0})$, we obtain $S_1 = \omega_1 t + S_{10}(R,Z;k_{Z0})$ where S_{10} is an integration constant. Then $k_{Z1} := \partial S_1/\partial Z = k_{Z11}t + k_{Z10}$ where $k_{Z11} := \partial\omega_1/\partial Z$ and $k_{Z10} := \partial S_{10}/\partial Z$. The wave number in the Z direction changes in time due to the Z dependence of ω_1 which originates from the density distribution in Z. Thus we can integrate Eq. (9) to obtain

$$\tilde{B}_R^{(0)}(R,Z,t;k_{Z0}) = \tilde{B}_R^{(0)}(R,Z,0;k_{Z0}) \times\exp\left[\frac{\mathrm{i}\,\bar{v}_\mathrm{A}^2 k_{Z0}}{\omega_1}\left(\frac{1}{2}\frac{\partial\omega_1}{\partial Z}t^2 + k_{Z10}t\right) \times\left\{1 \pm 2|\Omega|\left[4\bar{v}_\mathrm{A}^2 k_{Z0}^2 + (2\bar{\Omega} + R\bar{\Omega}')^2\right]^{-1/2}\right\}\right], \tag{10}$$

where $\tilde{B}_R^{(0)}(R,Z,0;k_{Z0})$ is an initial value. The term $(1/2)(\partial\omega_1/\partial Z)t^2$ in the exponential factor shows that the oscillation of the envelope becomes increasingly faster in time since it depends on t^2. The local frequency or the growth rate ω_1 depends on Z through the Alfvén velocity v_A or the density distribution (the magnetic field B is constant).

In conclusion, we have succeeded in capturing the transient phenomena of MRI, Eq. (10), for the first time; this has not been captured using the eigenvalue approach in long cylinder geometry. Such transient phenomena are absolutely necessary to explain the observed radiation variability [3]. Although we have used such a simple equilibrium magnetic field, it may be enough to point out the significant importance of the global treatment of MRI as well as the initial-value approach.

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