A Shell Model for the Hall MHD System

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A shell model has been formulated for the Hall MHD system, where the Hall term (scaled by the ion skin depth l_i) brings about a new singular perturbation in addition to the resistivity and viscosity effects. The model equations are derived under the constraint to conserve the energy and two helicities in the inertial range. In the limit of $l_i \to 0$, the system reduces to the conventional MHD shell model.

Keywords:
plasma turbulence, shell model, Hall MHD equations, Hall term, singular perturbation

So-called shell models have an advantage in understanding the universal dimensional characteristic of turbulence over a very wide range of length scales [1-4]. Compared to the standard neutral fluid turbulence, plasma turbulence may have much richer complexity due to its (i) multi-scale nature scaled by various scaling parameters, (ii) various instability drives, and (iii) strongly anisotropic nature of wave propagations (see Kraichnan's arguments on the Alfvénic turbulence [5]). To address these problems, plasma shell models need to have rather involved structures that must be formulated by some appropriate physical considerations. In this paper, we report our attempt to include, into the conventional MHD model [6-10], a new length scale, the ion skin depth characterizing the Hall effect in a high-temperature plasma. Three constants of motion (in the inertial range), viz., the energy, magnetic (mass-less electron) helicity, and ion helicity, suffice to determine the coefficients of the model equations.

The incompressible Hall MHD equations are:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} - \nabla_p + \nu\Delta\boldsymbol{u}, \tag{1}$$

$$\partial_t \boldsymbol{B} = \nabla \times [(\boldsymbol{u} - \varepsilon(\nabla \times \boldsymbol{B})) \times \boldsymbol{B}] + \eta\Delta\boldsymbol{B}, \tag{2}$$

with $\nabla\cdot\boldsymbol{u} = 0$, where $\boldsymbol{B}$ is the magnetic field (normalized by a representative value B_0), $\boldsymbol{u}$ is the velocity (normalized by the Alfvén speed $V_A = B_0/\sqrt{\mu_0 \rho M}$; M is the ion mass and the density ρ is assumed to be constant for simplicity), ν and η are the viscocity and the resistivity (normalized by $V_A L_0$; L_0 is a characteristic length scale) respectively, and p is the pressure (normalized by B_0^2/μ_0). The scale parameter $\varepsilon = l_i/L_0$ is called the Hall parameter, where $l_i = \sqrt{M/\mu_0 \rho e^2}$ is the ion skin depth. We note that the Hall MHD system has an intrinsic length scale l_i that scales the "singular perturbation" due to the Hall term. In the limit of $\varepsilon \to 0$, the Hall MHD system reduces to the conventional MHD system.

The ideal Hall MHD system ($\nu = 0$ and $\eta = 0$) has the following three constants of motion:

$$E = \frac{1}{2}\int(|\boldsymbol{u}|^2 + |\boldsymbol{B}|^2)\mathrm{d}x, \tag{3}$$

$$H_1 = \frac{1}{2}\int \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}x, \tag{4}$$

$$H_2 = \frac{1}{2}\int(\boldsymbol{A} + \varepsilon\boldsymbol{u})\cdot(\boldsymbol{B} + \varepsilon\nabla \times \boldsymbol{u})\mathrm{d}x, \tag{5}$$

where E is the energy, H_1 is the magnetic helicity, and H_2 is the ion helicity [11]. In the limit of $\varepsilon \to 0$, H_2 reduces to H_1. A modified ion helicity $(H_2 - H_1)/\varepsilon$, which is also a constant of motion of the Hall MHD system, reduces to the cross helicity as $\varepsilon \to 0$.

The shell model is constructed in a discrete scale hierarchy [2]. Each "shell" is characterized by a discrete parameter k_n that parallels the wave numbers (reciprocal length scales) of turbulent fluctuations in the corresponding scale hierarchy. To span a wide dynamic range of scales, we choose $k_n = k_0 q^n$ ($q = 2$, $1 \le n \le N$). We denote by u_n and B_n the complex-valued variables representing the flow and magnetic field in the n-th shell, respectively.

Essential inputs for formulating the evolution equation are:

1. The scaling and dimensionality of each term are consistent with those of the corresponding term in the original partial differential equations;
2. The nonlinear terms are represented by the quadratic nearest or second nearest neighborhood combinations of u_n and B_n;
3. The model equations have constants of motion that are commensurate with the constants of motion in the original partial differential equations.

In the shell model, the three constants of motion (3)-(5) read as:

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$$\hat{E} = \frac{1}{2}\sum_n (|B_n|^2 + |u_n|^2), \quad (6)$$

$$\hat{H}_1 = \frac{1}{2}\sum_n \frac{1}{k_n}(-1)^n |B_n|^2, \quad (7)$$

$$\hat{H}_2 = \frac{1}{2}\sum_n 2(B_n u_n^* + B_n^* u_n) + \varepsilon(-1)^n k_n |u_n|^2, \quad (8)$$

where $\hat{E}$, $\hat{H}_1$, and $\hat{H}_2$ parallel the energy, the magnetic helicity (the curl operator is expressed as $(-1)^n k_n$ in the shell model [10,12]), and the modified ion helicity $(H_2 - H_1)/\varepsilon = \frac{1}{2}\int \boldsymbol{A} \cdot (\nabla \times \boldsymbol{u}) + \boldsymbol{u} \cdot \boldsymbol{B} + \varepsilon \boldsymbol{u} \cdot (\nabla \times \boldsymbol{u}) \mathrm{d}\boldsymbol{x}$, respectively. The term $2(B_n u_n^* + B_n^* u_n)$ comes from the summation of $\boldsymbol{A} \cdot (\nabla \times \boldsymbol{u})$ and $\boldsymbol{u} \cdot \boldsymbol{B}$.

In formulating the Hall MHD shell model, we take quadratic combinations of the complex conjugate of u_n and B_n, and use the imaginary unit i in front of each term on the right hand side of the equation [2,10]. The Hall MHD shell model reads as:

$$\begin{aligned}\frac{\mathrm{d}u_n}{\mathrm{d}t} &= i\,[\alpha_n^{(1)} u_{n+2}^* u_{n+1}^* + \beta_n^{(1)} u_{n+1}^* u_{n-1}^* + \gamma_n^{(1)} u_{n-1}^* u_{n-2}^*] \\ &\quad -i\,[\alpha_n^{(1)} B_{n+2}^* B_{n+1}^* + \beta_n^{(1)} B_{n+1}^* B_{n-1}^* + \gamma_n^{(1)} B_{n-1}^* B_{n-2}^*] \\ &\quad - \nu k_n^2 u_n + f_n, \end{aligned} \quad (9)$$

$$\begin{aligned}\frac{\mathrm{d}B_n}{\mathrm{d}t} &= i\,[\alpha_n^{(2)} B_{n+2}^* u_{n+1}^* + \beta_n^{(2)} B_{n+1}^* u_{n-1}^* + \gamma_n^{(2)} B_{n-1}^* u_{n-2}^*] \\ &\quad -i\,[\alpha_n^{(2)} u_{n+2}^* B_{n+1}^* + \beta_n^{(2)} u_{n+1}^* B_{n-1}^* + \gamma_n^{(2)} u_{n-1}^* B_{n-2}^*] \\ &\quad -i\varepsilon[\alpha_n^{(3)} B_{n+2}^* B_{n+1}^* + \beta_n^{(3)} B_{n+1}^* B_{n-1}^* + \gamma_n^{(3)} B_{n-1}^* B_{n-2}^*] \\ &\quad -\eta k_n^2 B_n + g_n, \end{aligned} \quad (10)$$

where f_n and g_n denote the external force, and $\alpha_n^{(i)}$, $\beta_n^{(i)}$, $\gamma_n^{(i)}$ (i = 1, 2, 3) denote the coefficients.

For $\hat{E}$, $\hat{H}_1$, and $\hat{H}_2$ being constants of (9) and (10) in the inviscid and nonforcing limit, the coefficients $\alpha_n^{(i)}$, $\beta_n^{(i)}$, $\gamma_n^{(i)} t$ are determined as:

$$\alpha_n^{(1)} = k_n, \qquad \beta_n^{(1)} = -\tfrac{1}{2} k_{n-1}, \qquad \gamma_n^{(1)} = -\tfrac{1}{2} k_{n-2},$$

$$\alpha_n^{(2)} = \tfrac{1}{6} k_n, \qquad \beta_n^{(2)} = \tfrac{1}{3} k_{n-1}, \qquad \gamma_n^{(2)} = \tfrac{2}{3} k_{n-2},$$

$$\alpha_n^{(3)} = -\tfrac{1}{2}(-1)^n k_n^2, \qquad \beta_n^{(3)} = -\tfrac{1}{2}(-1)^{n-1} k_{n-1}^2,$$

$$\gamma_n^{(3)} = (-1)^{n-2} k_{n-2}^2,$$

$$\beta_1^{(1)} = \gamma_1^{(1)} = \gamma_2^{(1)} = \alpha_{N-1}^{(1)} = \alpha_N^{(1)} = \beta_N^{(1)} = 0,$$

$$\beta_1^{(2)} = \gamma_1^{(2)} = \gamma_2^{(2)} = \alpha_{N-1}^{(2)} = \alpha_N^{(2)} = \beta_N^{(2)} = 0,$$

$$\beta_1^{(3)} = \gamma_1^{(3)} = \gamma_2^{(3)} = \alpha_{N-1}^{(3)} = \alpha_N^{(3)} = \beta_N^{(3)} = 0.$$

In the limit of $\varepsilon \to 0$, the Hall MHD shell model is reduced to the conventional MHD shell model [10], and the modified ion helicity $\hat{H}_2$ of (8) also reduces to the cross helicity. Those facts are consistent with the relationship between the Hall MHD system and the conventional MHD system.

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